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ON PROBLEM FOR LINEAR PARABOLIC TYPE OF DIFFERENTIAL EQUATION WITH INTEGRAL CONDITIONS

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The finite difference method is used to solve one problem for a linear differential equation of parabolic type with integral conditions. A difference problem that approximates the original problem with the second order of accuracy is constructed, under certain conditions, the convergence of the method is proved and the rate of convergence is determined.

Keywords: differential equation, integral conditions, finite difference method, maximum principle, convergence
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Introduction

It is known that problems for equations of parabolic type are encountered in various areas of natural science. For example, processes associated with the distribution of heat, diffusion processes, etc. are described by parabolic type equations. In solving problems for these equations, in some cases, there are certain difficulties. Such difficulties also arise when the problem involves integral conditions. In this paper, we consider one problem for a parabolic differential equation with a changeable coefficient and integral conditions For a particular coefficient of the equation, integral conditions are determined, which are subsequently replaced by nonlocal boundary conditions. After such a replacement, the resulting new problem with non-local conditions is solved by the finite difference method.

Statement of the problem

The following problem for the parabolic type of equation is considered in present paper:

to find continuous function $u = u(x,t)$ in closed domain $\overline{D} = \{0 \le x \le l,\ 0 \le t \le T\}$, which satisfies to equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x} \right) + bu(x,t) + f(x,t),\ 0 < x < l,\ 0 < t \le T, \quad (1)$$

integral conditions

$$\int_0^l c_1(x)u(x,t)dx = \mu_1(t), \quad \int_0^l c_2(x)u(x,t)dx = \mu_2(t),\ 0 \le t \le T, \qquad (2)$$

and initial condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \qquad (3)$$

here $k(x) \ge k_0 > 0,\ f(x,t),\ c_1(x),\ c_2(x),\ \mu_1(t),\ \mu_2(t),\ \varphi(x)$ are known continuous functions of their arguments , $b-$ real number.

Integral conditions in (2) present certain difficulty in numerical solution of similar problems. In some papers, integral conditions of the form (2) for specific functions $c_1(x)$ and $c_2(x)$ are replaced by local boundary conditions (for example, [1]-[3]). This usually succeeds when the original equation is an equation with constant coefficients. In this paper, the given equation is an equation with changeable coefficients, and here, for arbitrary functions $k(x),\ c_1(x),\ c_2(x)$, it is difficult to replace conditions (2) by local boundary conditions. This work has set the following goal:

for a specific function $k(x) = (ax+d)^2,\ a > 0,\ d > 0,$, determine the class of functions $c_1(x),\ c_2(x)$, for which problem (1) - (3) can be reduced to a problem for equation (1) with local boundary conditions, construct a difference problem approximating it with the second order of accuracy, and investigate the convergence of the method.

Replacement of integral conditions

Suppose, $c(x)$ is arbitrary function . Consider condition

$$\int_0^l c(x)u(x,t)dx = \mu(t).$$

Differentiating the last with respect to t, we'll get:

$$\int_0^l c(x)\frac{\partial u(x,t)}{\partial t}dx = \mu'(t).$$

Due to equation (1) with $k(x) = (ax+d)^2$ we have:

$$\int_0^l c(x)\left[\frac{\partial}{\partial x}\left((ax+d)^2\frac{\partial u(x,t)}{\partial x}\right) + bu(x,t) + f(x,t)\right]dx = \mu'(t)$$

or

$$\int_0^l c(x)\frac{\partial}{\partial x}\left((ax+d)^2\frac{\partial u(x,t)}{\partial x}\right)dx = \mu'(t) - b\mu(t) - \int_0^l c(x)f(x,t)dx. \qquad (4)$$

Applying twice the formula of integration by parts to the integral on the left-hand side of this equality, after elementary transformations, we'll get:

$$c(l)(al+d)^2\frac{\partial u(l,t)}{\partial x} - c(0)b^2\frac{\partial u(0,t)}{\partial x} - c'(l)(al+d)^2u(l,t) + c'(0)d^2u(0,t) +$$

$$+\int_0^l \left(c'(x)(ax+d)^2\right)' u(x,t)dx = \mu'(t) - b\mu(t) - \int_0^l c(x)f(x,t)dx. \quad (5)$$

Let the function $c(x)$ be determined from the condition

$$\left(c'(x)(ax+d)^2\right)' = \alpha \cdot c(x), \quad (6)$$

where α – real number. This equality defines the Euler equation with respect to a function $c(x)$, of the form

$$(ax+d)^2 c''(x) + 2a(ax+d)c'(x) - \alpha c(x) = 0. \quad (7)$$

This equation can easily be reduced to a linear differential equation with constant coefficients by substitution $ax+d=e^t$. After such a change, equation (7) takes the form

$$a^2 c''(t) + a^2 c'(t) - \alpha \cdot c(t) = 0. \quad (8)$$

The characteristic equation corresponding to this equation has the form

$$a^2k^2 + a^2k - \alpha = 0.$$

The roots of this equation are determined by the equalities

$$k_1 = \frac{-a+\sqrt{a^2+4\alpha}}{2a}, \; k_2 = \frac{-a-\sqrt{a^2+4\alpha}}{2a}.$$

If $a^2+4\alpha>0$, then linearly independent solutions of equation (7) will be determined by the equalities

$$c_1(x) = (ax+d)^{k_1}, \; c_2(x) = (ax+d)^{k_2}. \quad (9)$$

If $a^2+4\alpha=0$, then linearly independent solutions takes the form

$$c_1(x) = \frac{1}{\sqrt{ax+d}}, \; c_2(x) = \frac{1}{\sqrt{ax+d}}\ln(ax+d), \quad (10)$$

but, if $a^2+4\alpha<0$, then in this case we have

$$c_1(x) = \frac{1}{\sqrt{ax+d}}\cos\left(\frac{\sqrt{-a^2-4\alpha}}{2a}\ln(ax+d)\right), c_2(x) = \frac{1}{\sqrt{ax+d}}\sin\left(\frac{\sqrt{-a^2-4\alpha}}{2a}\ln(ax+d)\right). \quad (11)$$

Hence it follows that if the functions $c_1(x)$ and $c_2(x)$ are defined by equalities in (9) or in (10), or in (11), then for these functions equality (6) will hold and equality (5) will take the form

$$c(l)(al+d)^2\frac{\partial u(l,t)}{\partial x} - c(0)d^2\frac{\partial u(0,t)}{\partial x} - c'(l)(al+d)^2u(l,t) + c'(0)d^2u(0,t) =$$

$$= \mu'(t) - (b+\alpha)\mu(t) - \int_0^l c(x)f(x,t)dx. \quad (12)$$

Therefore, if the functions $c_1(x)$ and $c_2(x)$ in (2) are determined by equalities (9) or (10) or (11), then the integral conditions (2) can be replaced by the following boundary conditions:

$$c_1(l)(al+d)^2\frac{\partial u(l,t)}{\partial x}-c_1(0)d^2\frac{\partial u(0,t)}{\partial x}-c_1'(l)(al+d)^2u(l,t)+c_1'(0)d^2u(0,t)=$$

$$=\mu_1'(t)-(b+\alpha)\mu_1(t)-\int_0^l c_1(x)f(x,t)dx. \quad (13)$$

$$c_2(l)(al+d)^2\frac{\partial u(l,t)}{\partial x}-c_2(0)d^2\frac{\partial u(0,t)}{\partial x}-c_2'(l)(al+d)^2u(l,t)+c_2'(0)d^2u(0,t)=$$

$$=\mu_2'(t)-(b+\alpha)\mu_2(t)-\int_0^l c_2(x)f(x,t)dx. \quad (14)$$

If we exclude from the last equalities, first $\frac{\partial u(l,t)}{\partial x}$, then $\frac{\partial u(0,t)}{\partial x}$, then instead of these conditions we get the conditions

$$\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)d^2\frac{\partial u(0,t)}{\partial x}-\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)d^2u(0,t)+$$

$$+\left(c_1'(l)c_2(l)-c_1(l)c_2'(l)\right)(al+d)^2u(l,t)=\bar{\mu}_1(t), \quad (15)$$

$$\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)(al+d)^2\frac{\partial u(l,t)}{\partial x}-\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2u(0,t)+$$

$$+\left(c_1'(l)c_2(0)-c_1(0)c_2'(l)\right)(al+d)^2u(l,t)=\bar{\mu}_2(t), \quad (16)$$

where

$$\bar{\mu}_1(t)=c_1(l)\mu_2'(t)-c_2(l)\mu_1'(t)+(b+\alpha)\left(c_2(l)\mu_1(t)-c_1(l)\mu_2(t)\right)+$$
$$+c_2(l)\int_0^l c_1(x)f(x,t)dx-c_1(l)\int_0^l c_2(x)f(x,t)dx,$$

$$\bar{\mu}_2(t)=c_1(0)\mu_2'(t)-c_2(0)\mu_1'(t)+(b+\alpha)\left(c_2(0)\mu_1(t)-c_1(0)\mu_2(t)\right)+$$
$$+c_2(0)\int_0^l c_1(x)f(x,t)dx-c_1(0)\int_0^l c_2(x)f(x,t)dx.$$

It is easy to check that in all three cases the functions $c_1(x)$ and $c_2(x)$ satisfy the condition $c_1(0)c_2(l)-c_1(l)c_2(0)\neq 0$. Therefore, dividing both sides of equalities (15) and (16) by this expression, conditions (15) and (16) can be rewritten in the form

$$d^2\frac{\partial u(0,t)}{\partial x}-\frac{\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(0,t)+\frac{\left(c_1'(l)c_2(l)-c_1(l)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(l,t)=\tilde{\mu}_1(t) \quad (17)$$

$$(al+d)^2\frac{\partial u(l,t)}{\partial x}-\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(0,t)+$$
$$+\frac{\left(c_1'(l)c_2(0)-c_1(0)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(l,t)=\tilde{\mu}_2(t), \quad (18)$$

where

$$\tilde{\mu}_1(t)=\frac{\overline{\mu}_1(t)}{c_1(0)c_2(l)-c_1(l)c_2(0)}, \quad \tilde{\mu}_2(t)=\frac{\overline{\mu}_2(t)}{c_1(0)c_2(l)-c_1(l)c_2(0)}.$$

Using Taylor's formula, we get:

$$k(x)\frac{\partial u(x,t)}{\partial x}=k(0)\frac{\partial u(0,t)}{\partial x}+x\left[\frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right)\right]_{x=0}+O\left(x^2\right).$$

Taking $x=\frac{h}{2}$, in the last equality, we can easily obtain the equality

$$k(0)\frac{\partial u(0,t)}{\partial x}=k\left(\frac{h}{2}\right)\frac{u(x_1,t)-u(0,t)}{h}-\frac{h}{2}\left[\frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right)\right]_{x=0}+O\left(h^2\right).$$

Similarly, we obtain the validity of the equality

$$k(l)\frac{\partial u(l,t)}{\partial x}=k\left(l-\frac{h}{2}\right)\frac{u(x_N,t)-u(x_{N-1},t)}{h}+\frac{h}{2}\left[\frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right)\right]_{x=l}+O\left(h^2\right).$$

Assuming the fulfillment of equation (1) on the boundaries $x=0$ and $x=l$ of the domain $\overline{D}$, from the last two equalities, taking into account, that $k(x)=(ax+d)^2$, we obtain :

$$d^2\frac{\partial u(0,t)}{\partial x}=\left(\frac{ah}{2}+d\right)^2\frac{u(x_1,t)-u(0,t)}{h}-\frac{h}{2}\left[\frac{\partial u(0,t)}{\partial t}-bu(0,t)-f(0,t)\right]+O\left(h^2\right),$$

$$(al+d)^2\frac{\partial u(l,t)}{\partial x}=\left(al-\frac{ah}{2}+d\right)^2\frac{u(x_N,t)-u(x_{N-1},t)}{h}+\frac{h}{2}\left[\frac{\partial u(l,t)}{\partial t}-bu(l,t)-f(l,t)\right]+O\left(h^2\right).$$

Due to these equalities, conditions (17) - (18) can be rewritten in the form

$$\left(\frac{ah}{2}+d\right)^2\frac{u(x_1,t)-u(0,t)}{h}-\frac{h}{2}\left[\frac{\partial u(0,t)}{\partial t}-bu(0,t)\right]-\frac{\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\cdot u(0,t)+$$
$$+\frac{\left(c_1'(l)c_2(l)-c_1(l)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(l,t)=\tilde{\mu}_1(t)-\frac{h}{2}f(0,t)+O\left(h^2\right), \qquad (19)$$

$$\left(al-\frac{ah}{2}+d\right)^2\frac{u(x_N,t)-u(x_{N-1},t)}{h}+\frac{h}{2}\left[\frac{\partial u(l,t)}{\partial t}-bu(l,t)\right]-\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\times$$
$$\times u(0,t)+\frac{\left(c_1'(l)c_2(0)-c_1(0)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(l,t)=\tilde{\mu}_2(t)+\frac{h}{2}f(l,t)+O\left(h^2\right). \qquad (20)$$

Difference problem

Divide the segment $[0,\ l\,]$ of the axis Ox by points $x_n=nh,\quad n=0,1,2,\ldots,N,$ $h=l/N$, into N equal parts, and the segment $[0,T\,]$ of the Ot axis by points $t_j=j\tau,\ j=0,1,2,\ldots,j_0,\ \tau=T/j_0$, into j_0, equal parts.

Define in the domain $\overline{D}$ a grid $\overline{\omega}_{h\tau}=\left\{(x_n,t_j),\ n=0,1,2,\ldots,N,\ j=0,1\ldots,j_0\right\}$.

Using, instead of integral conditions (2), the last equalities (19) and (20), to problem (1) - (3) on the grid $\overline{\omega}_{h\tau}$ we can associate the following difference problem :

$$\frac{h}{2}\frac{y_0^{j+1}-y_0^j}{\tau}-\frac{1}{2}\left(\frac{ah}{2}+d\right)^2\left(\frac{y_1^{j+1}-y_0^{j+1}}{h}+\frac{y_1^j-y_0^j}{h}\right)-\frac{bh}{2}\frac{y_0^{j+1}+y_0^j}{2}+$$
$$+\frac{\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_0^{j+1}+y_0^j}{2}-\frac{\left(c_1'(l)c_2(l)-c_1(l)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_N^{j+1}+y_N^j}{2}=f_0^j,$$

$$\frac{y_n^{j+1}-y_n^j}{\tau}-\frac{1}{2}\left(ax_n+\frac{ah}{2}+d\right)^2\frac{y_{n+1}^{j+1}-y_n^{j+1}+y_{n+1}^j-y_n^j}{h^2}+\frac{1}{2}\left(ax_n-\frac{ah}{2}+d\right)^2\cdot$$
$$\cdot\frac{y_n^{j+1}-y_{n-1}^{j+1}+y_n^j-y_{n-1}^j}{h^2}-b\frac{y_n^{j+1}+y_n^j}{2}=f_n^j,\quad n=1,2,\ldots,N-1, \qquad (21)$$

$$\frac{h}{2}\frac{y_N^{j+1}-y_N^j}{\tau}+\frac{1}{2}\left(al-\frac{ah}{2}+d\right)^2\left(\frac{y_N^{j+1}-y_{N-1}^{j+1}}{h}+\frac{y_N^j-y_{N-1}^j}{h}\right)-\frac{bh}{2}\frac{y_N^{j+1}+y_N^j}{2}-$$

$$-\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_0^{j+1}+y_0^j}{2}+\frac{\left(c_1'(l)c_2(0)-c_1(0)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_N^{j+1}+y_N^j}{2}=f_N^j,$$

$$j=0,1,\ldots,\ j_0-1,$$

$$y_n^0=\varphi(x_n),\ n=0,1,\ldots,N. \qquad (22)$$

where

$$f_0^j=-\tilde{\mu}_1\left(t_j+0,5\tau\right)+\frac{h}{2}f\left(0,t_j+0,5\tau\right),\ f_N^j=\tilde{\mu}_2\left(t_j+0,5\tau\right)+\frac{h}{2}f\left(l,t_j+0,5\tau\right),$$

$$f_n^j=f\left(x_n,t_j+0,5\tau\right),\ n=1,2,\ldots,N-1.$$

It should be noted that the difference problem (21) - (22) approximates problem (1) - (3) with an accuracy $O\left(h^2+\tau^2\right)$, if the solution $u(x,t)$ of problem (1) - (3) has bounded partial derivatives into domain $\overline{D}$ with respect to the variable x up to the fourth order , and with respect to the variable t up to the third order and equation (1) is fulfilled both on the boundaries $x=0$ and $x=l$ of domain $\overline{D}$.

Difference problem (21) - (22), for each value j, there is a three-point difference problem. It can be solved, for example, by the well-known sweep method for such difference problems.

Maximum Principle and Consequence of the Maximum Principle

Consider the difference problem (21) - (22) and prove the validity of the following theorem (maximum principle) with respect to the solution of this problem.

Theorem 1 (Maximum principle). Let the grid function
$y_n^j, n=0,1,\ldots,N, j=0,1,\ldots,j_0,$
satisfies problem (21) - (22). Let the conditions
$f_n^j\le 0\ (f_n^j\ge 0), n=0,1,\ldots,N,\ \ j=0,1,\ldots,\ j_0-1$ are satisfies. If

$$b\le 0,\ \frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\ge 0,\ \frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\ge 0,$$

$$\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}d^2-\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}(al+d)^2\ge\varepsilon>0,$$

$$\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}(al+d)^2-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}d^2\ge\delta>0, \qquad (23)$$

$$\tau \le \min\left\{ \frac{2h^2}{2(al+d)^2 - bh^2},\ h^2\left[d^2 - \frac{bh^2}{2} + hd^2\,\frac{c_1'(0)c_2(l) - c_1(l)c_2'(0)}{c_1(0)c_2(l) - c_1(l)c_2(0)} \right]^{-1},\right.$$

$$\left. h^2\left[d^2 - \frac{bh^2}{2} + h(al+d)^2\,\frac{c_1'(l)c_2(0) - c_1(0)c_2'(l)}{c_1(0)c_2(l) - c_1(l)c_2(0)} \right]^{-1} \right\},$$

Then solution $y_n^j, n = 0,1,\ldots,N,\ j = 0,1,\ldots,j_0,$ of the problem (21)-(22), differ from constant , cannot take the largest positive (smallest negative) value at $n = 0,1,\ldots, N,\ j = 1,2,\ldots,\ j_0$.

P r o o f. Let prove the first part of the theorem. Suppose $f_n^j \le 0, n = 0,1,\ldots,N,\ j = 0,1,\ldots,\ j_0 - 1,$ and conditions (23) are satisfied, but the solution y_n^j of the problem (21) - (22) takes the largest positive value at $n = n_0,\ j = i+1,\ 0 \le n_0 \le N,\ 0 \le i \le j_0 - 1:$

$$y_{n_0}^{i+1} = \max_{0\le n\le N,\ 0\le j\le j_0} y_n^j = M > 0.$$

Suppose, that $0 < n_0 < N.$ Without loss of generality, we can assume that $y_{n_0}^{i+1} > y_{n_0-1}^{i+1}$. Let’s consider the difference equation in (21) at $n = n_0,\ j = i:$

$$f_{n_0}^i = -\frac{1}{2h^2}\left(ax_{n_0} - \frac{ah}{2} + d\right)^2 y_{n_0-1}^{i+1} + \left(\frac{1}{\tau} + \frac{1}{2h^2}\left(ax_{n_0} + \frac{ah}{2} + d\right)^2 + \frac{1}{2h^2}\left(ax_{n_0} - \frac{ah}{2} + d\right)^2 - \frac{b}{2}\right) y_{n_0}^{i+1} -$$

$$- \frac{1}{2h^2}\left(ax_{n_0} + \frac{ah}{2} + d\right)^2 y_{n_0+1}^{i+1} - \frac{1}{2h^2}\left(ax_{n_0} - \frac{ah}{2} + d\right)^2 y_{n_0-1}^{i} + \left(-\frac{1}{\tau} + \frac{1}{2h^2}\left(ax_{n_0} + \frac{ah}{2} + d\right)^2 +\right.$$

$$\left. + \frac{1}{2h^2}\left(ax_{n_0} - \frac{ah}{2} + d\right)^2 - \frac{b}{2}\right)\cdot y_{n_0}^{i} - \frac{1}{2h^2}\left(ax_{n_0} + \frac{ah}{2} + d\right)^2 y_{n_0+1}^{i} > -bM \ge 0,$$

since by the hypothesis of the theorem $b \le 0.$ This contradicts the condition $f_{n_0}^i \le 0.$

Suppose, that $n_0 = 0.$ Without loss of generality, we can assume that $y_0^{i+1} > y_1^{i+1}$. Consider the first equation in (21) at $j = i:$

$$f_0^i = \left(\frac{h}{2\tau} + \frac{1}{2h}\left(\frac{ah}{2} + d\right)^2 - \frac{bh}{4} + \frac{d^2}{2}\,\frac{c_1'(0)c_2(l) - c_1(l)c_2'(0)}{c_1(0)c_2(l) - c_1(l)c_2(0)}\right) y_0^{i+1} - \left(\frac{h}{2\tau} - \frac{1}{2h}\left(\frac{ah}{2} + d\right)^2 +\right.$$

$$+\frac{bh}{4}-\frac{d^2}{2}\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\Bigg)y_0^i-\frac{1}{2h}\left(\frac{ah}{2}+d\right)^2 y_1^{i+1}-\frac{1}{2h}\left(\frac{ah}{2}+d\right)^2 y_1^i-$$

$$-\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}(al+d)^2\frac{y_N^{i+1}+y_N^i}{2}>\left(\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}d^2-\right.$$

$$\left.-\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}(al+d)^2\right)M\ge\varepsilon M>0.$$

This contradicts the condition $f_0^i\le 0$.

Suppose, that $n_0=N$. Without loss of generality, we can assume that $y_N^{i+1}>y_{N-1}^{i+1}$. Consider the last equation in (21) at $j=i$:

$$f_N^i=\left(\frac{h}{2\tau}+\frac{1}{2h}k\left(l-\frac{h}{2}\right)-\frac{bh}{4}+\frac{k(l)}{2}\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right)y_N^{i+1}-\left(\frac{h}{2\tau}-\frac{1}{2h}k\left(l-\frac{h}{2}\right)+\right.$$

$$\left.+\frac{bh}{4}-\frac{k(l)}{2}\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right)y_N^i-\frac{1}{2h}k\left(l-\frac{h}{2}\right)y_{N-1}^{i+1}-\frac{1}{2h}k\left(l-\frac{h}{2}\right)y_{N-1}^i-$$

$$-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}d^2\frac{y_0^{i+1}+y_0^i}{2}>\left(\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}(al+d)^2-\right.$$

$$\left.-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}d^2\right)M\ge\delta M>0.$$

The first part of the theorem is proved. The second part of the theorem can be proved in a similar way.

Theorem 2. Let the grid function $y_n^j, n=0,1,\ldots,N,\ j=0,1,\ldots j_0$, satisfies to problem (21)-(22). If $f_n^j\le 0,\ \varphi(x_n)\le 0\left(f_n^j\ge 0,\ \varphi(x_n)\ge 0\right),\ n=0,1,\ldots,N,$ $j=0,1,\ldots,\ j_0-1,$ and conditions (23) are satisfied, then $y_n^j\le 0$ $\left(y_n^j\ge 0\right), n=0,1,\ldots,N,\quad j=0,1,\ldots j_0.$

The validity of the statement of this theorem follows from the maximum principle.

Corollary. Suppose, that conditions (23) are satisfied. Then the homogeneous problem corresponding to problem (21) - (22) has only the trivial solution

$$y_n^j=0,\ n=0,1,\ldots,N,\quad j=0,1,\ldots,\ j_0.$$

It follows from this corollary that under conditions (23) there is a unique solution to the difference problem (21) - (22).

Theorem 3 (Comparison theorem). Suppose, that $y_n^j, n=0,1,\ldots,N,$

$j=0,1,\dots j_0-$ solution of the difference problem (21)-(22) and $\tilde{y}_n^j, n=0,1,\dots,N,$ $j=0,1,\dots j_0-$ solution of the difference problem obtained by replacing in (21)-(22) the functions $f_n^j,\ n=0,1,\dots,N,\ j=0,1,2,\dots,j_0-1,$ and $\varphi(x_n),$ $n=0,1,\dots,N,$ correspondingly, by $\tilde{f}_n^j, n=0,1,\dots,N, j=0,1,2,\dots,j_0-1,$ and $\tilde{\varphi}(x_n),\ n=0,1,\dots,N.$ Then, if $\left|f_n^j\right|\le \tilde{f}_n^j,\ n=0,1,\dots,N,\ j=0,1,\dots,j_0-1,$ and $\left|\varphi(x_n)\right|\le\tilde{\varphi}(x_n),$ $n=0,1,\dots,N,$ then under conditions (23) inequalities $\left|y_n^j\right|\le\tilde{y}_n^j,\quad n=0,1\dots,N,\ j=0,1,\dots,j_0$ take place.

Convergence

In the grid domain , $\overline{\omega}_{h\tau}$ we define the grid function z_n^j by the equality

$$z_n^j=y_n^j-u(x_n,t_j),\ n=0,1,\dots,N,\ j=0,1,\dots,j_0,$$

where y_n^j - solution of the difference problem (21)-(22), $u(x_n,t_j)$- the value of the exact solution to problem (1) - (3) at the grid point (x_n,t_j) of grid $\overline{\omega}_{h\tau}$. If we substitute the expression y_n^j found from the last equality in the difference problem (21) - (22), then with respect to the function z_n^j we obtain:

$$\frac{h}{2}\frac{z_0^{j+1}-z_0^j}{\tau}-\frac{1}{2}\left(\frac{ah}{2}+d\right)^2\left(\frac{z_1^{j+1}-z_0^{j+1}}{h}+\frac{z_1^j-z_0^j}{h}\right)-\frac{bh}{2}\frac{z_0^{j+1}+z_0^j}{2}+$$

$$+\frac{\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{z_0^{j+1}+z_0^j}{2}-\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{z_N^{j+1}+z_N^j}{2}=\psi_0^j,$$

$$\frac{z_n^{j+1}-z_n^j}{\tau}-\frac{1}{2}\left(ax_n+\frac{ah}{2}+d\right)^2\frac{z_{n+1}^{j+1}-z_n^{j+1}+z_{n+1}^j-z_n^j}{h^2}+\frac{1}{2}\left(ax_n-\frac{ah}{2}+d\right)^2\cdot$$

$$\cdot\frac{z_n^{j+1}-z_{n-1}^{j+1}+z_n^j-z_{n-1}^j}{h^2}-b\frac{z_n^{j+1}+z_n^j}{2}=\psi_n^j,\ n=1,2,\dots,N-1, \tag{24}$$

$$\frac{h}{2}\frac{z_N^{j+1}-z_N^j}{\tau}+\frac{1}{2}\left(al-\frac{ah}{2}+d\right)^2\left(\frac{z_N^{j+1}-z_{N-1}^{j+1}}{h}+\frac{z_N^j-z_{N-1}^j}{h}\right)-\frac{bh}{2}\frac{z_N^{j+1}+z_N^j}{2}-$$

$$-\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{z_0^{j+1}+z_0^j}{2}+\frac{\left(c_1'(l)c_2(0)-c_1(0)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\times$$

$$\times\frac{z_N^{j+1}+z_N^j}{2}==\psi_N^j,\qquad j=0,1,\dots,j_0-1, \tag{25}$$

$$z_n^0=0,\ \ n=0,1,\dots,N,$$

here $\psi_n^j, n=0,1,\dots,N-$ determine the error of approximation of the difference problem (21) - (22).

The right-hand sides of difference equations (24) satisfy the estimate

$$\left|\psi_n^j\right|\le L\left(h^2+\tau^2\right),\ n=0,1,\dots,N,\ j=0,1,\dots,j_0-1,$$

where $L>0-$ some constant.

Let's define the grid function on the grid $\overline{\omega}_{h\tau}$

$$\widetilde{z}_n^j=L\xi\left(h^2+\tau^2\right)(2l_1-x_n),\ n=0,1,\dots,N,\ j=0,1,\dots,j_0, \tag{26}$$

where $\xi>0$, $l_1\ge l$ are constants. Obviously, $\widetilde{z}_n^j$ is a positive function. For this function, under conditions (23), after simple transformations we have:

$$\frac{h}{2}\frac{\widetilde{z}_0^{j+1}-\widetilde{z}_0^j}{\tau}-\frac{1}{2}\left(\frac{ah}{2}+d\right)^2\left(\frac{\widetilde{z}_1^{j+1}-\widetilde{z}_0^{j+1}}{h}+\frac{\widetilde{z}_1^j-\widetilde{z}_0^j}{h}\right)-\frac{bh}{2}\frac{\widetilde{z}_0^{j+1}+\widetilde{z}_0^j}{2}+$$

$$+\frac{\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{\widetilde{z}_0^{j+1}+\widetilde{z}_0^j}{2}-\frac{\left(c_1'(l)c_2(l)-c_1(l)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{\widetilde{z}_N^{j+1}+\widetilde{z}_N^j}{2}=$$

$$=L\xi\left(h^2+\tau^2\right)\left[\left(\frac{ah}{2}+d\right)^2+\frac{\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\cdot 2l_1-\right.$$

$$\left.-\frac{\left(c_1'(l)c_2(l)-c_1(l)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\cdot(2l_1-l)\right]>L\xi\left(h^2+\tau^2\right)\left[\left(\frac{ah}{2}+d\right)^2+\right.$$

$$\left.+\left(\frac{\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}-\frac{\left(c_1'(l)c_2(l)-c_1(l)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right)\cdot 2l_1\right]\ge L\xi\left(h^2+\tau^2\right)\cdot$$

$$\cdot\left(d^2+2\varepsilon l_1\right)\ge L\left(h^2+\tau^2\right), \tag{27}$$

if $\xi\ge\dfrac{1}{d^2+2\varepsilon l_1}$.

$$\frac{\widetilde{z}_n^{j+1}-\widetilde{z}_n^j}{\tau}-\frac{1}{2}\left(ax_n+\frac{ah}{2}+d\right)^2\frac{\widetilde{z}_{n+1}^{j+1}-\widetilde{z}_n^{j+1}+\widetilde{z}_{n+1}^j-\widetilde{z}_n^j}{h^2}+\frac{1}{2}\left(ax_n-\frac{ah}{2}+d\right)^2\cdot$$

$$\cdot\frac{\widetilde{z}_n^{j+1}-\widetilde{z}_{n-1}^{j+1}+\widetilde{z}_n^{j}-\widetilde{z}_{n-1}^{j}}{h^2}-b\frac{\widetilde{z}_n^{j+1}+\widetilde{z}_n^{j}}{2}=L\xi\left(h^2+\tau^2\right)\cdot[2a(ax_n+d)-b(2l_1-x_n)]\geq$$

$$\geq L\xi\left(h^2+\tau^2\right)\cdot 2(ad-bl_1)\geq L\left(h^2+\tau^2\right),\qquad n=1,2,\dots,N-1, \tag{28}$$

if $\xi\geq\dfrac{1}{2(ad-bl_1)}$.

$$\frac{h}{2}\frac{\widetilde{z}_N^{j+1}-\widetilde{z}_N^{j}}{\tau}+\frac{1}{2}\left(al-\frac{ah}{2}+d\right)^2\left(\frac{\widetilde{z}_N^{j+1}-\widetilde{z}_{N-1}^{j+1}}{h}+\frac{\widetilde{z}_N^{j}-\widetilde{z}_{N-1}^{j}}{h}\right)-\frac{bh}{2}\frac{\widetilde{z}_N^{j+1}+\widetilde{z}_N^{j}}{2}-$$

$$-\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{\widetilde{z}_0^{j+1}+\widetilde{z}_0^{j}}{2}+\frac{\left(c_1'(l)c_2(0)-c_1(0)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{\widetilde{z}_N^{j+1}+\widetilde{z}_N^{j}}{2}=$$

$$=L\xi\left(h^2+\tau^2\right)\left[-2\left(al-\frac{ah}{2}+d\right)^2-\frac{bh}{2}(2l_1-l)+\left(\frac{\left(c_1'(l)c_2(0)-c_1(0)c_2'(l)\right)(al+d)^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}-\right.\right.$$

$$\left.\left.-\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right)\cdot(2l_1-l)-\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\cdot l\right]\geq$$

$$\geq L\xi\left(h^2+\tau^2\right)\left[-2\left(al-\frac{ah}{2}+d\right)^2-\frac{bh}{2}(2l_1-l)+\delta(2l_1-l)-\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\cdot l\right]\geq$$

$$\geq L\left(h^2+\tau^2\right), \tag{29}$$

if

$$l_1\geq l_2=\frac{l}{2}+\frac{1}{2\delta}\left[1+2\left(al-\frac{ah}{2}+d\right)^2+\frac{\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)d^2}{c_1(0)c_2(l)-c_1(l)c_2(0)}\cdot l\right]$$

and $\xi\geq 1$.

Thus, if ξ in (26) we define from the condition

$$\xi=\max\left(1,\frac{1}{d^2+2\varepsilon l_1},\ \frac{1}{2(ad-bl_1)}\right), \tag{30}$$

and l_1 from the condition

$$l_1=\max\left(l,l_2\right), \tag{31}$$

then for function $\widetilde{z}_n^{j}$, defined by equality (26), inequalities (27) - (29) will be satisfies and

$$\tilde{z}_n^0 = L\xi\left(h^2+\tau^2\right)(2l_1 - x_n),\ n=0,1,\ldots,N,\ j=0,1,\ldots,j_0. \qquad (32)$$

Therefore, comparing problem (24) - (25) with problem (27) - (29), (32), by virtue of the comparison theorem, we obtain the inequality

$$\left|z_n^j\right| \le \tilde{z}_n^j,\quad n=0,1,\ldots,N,\ j=0,1,2,\ldots,j_0,$$

or

$$\left|y_n^j - u(x_n,t_j)\right| \le L\xi\left(h^2+\tau^2\right)\cdot 2l_1,\ n=0,1,\ldots,N,\ j=0,1,\ldots,j_0. \qquad (33)$$

So, the following holds:

Theorem 4. Let the solution of equation (1) in domain $D=\{0<x<l,\ 0<t\le T\}$ have bounded partial derivatives with respect to variable x up to the fourth order and with respect to t up to the third order, and this equation is fulfilled both on the boundaries $x=0$ and $x=l$ on the domain $\overline{D}$. If conditions (23) are satisfied, then the solution of the difference problem (21) - (22) converges to the solution of the problem (1) - (3). Moreover, estimate (33) holds.

Conclusions

The problem for an equation of parabolic type with specific integral conditions is reduced to the problem for this equation with nonlocal boundary conditions, and then the method of finite differences is applied to the solution of the new problem obtained. A difference problem that approximates this problem with the second order of accuracy is constructed. Under certain conditions, the maximum principle and some other theorems regarding the solution of the difference problem are proved. Using these theorems, the convergence of the solution of a difference problem to the solution of a problem with non-local boundary conditions is proved and an estimate for the rate of convergence is obtained.

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THEORETICAL FOUNDATIONS OF ARTIFICIAL NEURAL NETWORKS

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A neural network is a sequence of neurons interconnected by synapses. The structure of the neural network came to the world of programming straight from biology. Thanks to this structure, the machine acquires the ability to analyze and even memorize various information. Neural networks are also capable of not only analyzing incoming information, but also reproducing it from their memory.

Keywords: Neural networks; Artificial intelligence;

Neural networks are used to solve complex problems that require analytical calculations, similar to what the human brain does. The most common applications of neural networks are:

– Classification;

– Prediction;

- Recognition.

Classification is the distribution of data by parameters. For example, at the input a set of people is given and it is necessary to decide which of them to give a loan and who not. This work can be done using a neural network that analyzes information such as: age, solvency, credit history, and so on.

Prediction is the ability to predict the next step. For example, the growth or fall of shares, depending on the situation on the stock market.

Recognition is currently the widest application of neural networks. Used by Google when you search for a photo or in phone cameras when it detects the position of your face and highlights it and more.

Neural network technologies are used to solve such problems, in which there are no clear algorithms for obtaining the desired results. Many huge corporations are interested in neural networks.

Benefits of neural networks:

– Opportunity to learn from many examples in cases where the laws governing the development of the situation and the dependence function between input and output data are not known;

– Ability to successfully solve problems based on incomplete, distorted and internally contradictory input information;

– The ability to operate a trained neural network with any users;

- The ability to extremely easily connect neural network packages to databases, e-mail and automate the process of entering and primary data processing;

A neural network is a handy tool that allows you to determine the nature of the influence of a related scorecard. Having studied the article, we can conclude that the use of neural networks can provide a significant gain compared to linear classifier models. Artificial neural networks are the most promising method for assessing the solvency of borrowers, as they are able to take into account a large number of economic characteristics, as well as independently identify the most complex relationships between them, without using complex computing resources in hardware implementation. However, the general limitation of the predictive ability of such models is a consequence of their static nature - the data used refer to the same time period.

Conclusion

In conclusion, we can say that the introduction of neural networks in the implementation of decision support systems seems to be a promising direction. The use of such systems in the economy has no restrictions. Already now there are decision support systems based on neural network technologies used by the largest foreign companies in order to reduce risks in planning their financial activities.

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COMPLEX FUZZY EVALUATIONS OF SOCIAL CAPITAL

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In this paper with the purpose to establish the level of social capital (SC) in the country, two controversial methodologies of preference of expert opinions (PEO) and intuitionistic linguistic assessment of SC are proposed. The Z-number theory facilitated new possibilities to research more kinds of uncertainty problems, especially in the areas of multiple criteria decision making and events stated in natural language. In the first methodology expert opinions given in Z-numbers are fuzzified, then employing average graded general representation technique, fuzzified values are converted into crisp values, finally their priorities are established. In comparison to the first methodology, in which the core approach is the conversion of Z-numbers into crisp data, in the second methodology expert opinions given in crisp values are converted into Z-numbers. The generation of z-numbers is a practical instrument to make choices among expert evaluations. In the second methodology with the intention to make PEO, the experimental researches on modified generation of Z-numbers based on ordered weighted average and maximum entropy is used. As a result, with the PEO methodology in two controvertible ways where preferred expert opinions are obtained. Thus, inferred results from intuitionistic linguistic aggregation of SC illustrate the level of SC in Azerbaijan.

Keywords: social capital, expert opinions, Z-numbers, maximum entropy, average graded general representation, priority weights.
UDC Numbers: 330.4, 338.2

Introduction

Social capital-made up of social obligations “connections”, which is convertible, in certain conditions, into economic capital and may be institutionalized in the form of a title of nobility.

Social capital is the aggregate of the actual or potential resources which are linked to possession of a durable network of more or less institutionalized relationships of mutual acquaintance and recognition – or in other words, to membership in a group-which provides each of its members with the backing of the collectively- owned capital, a “credential” which entitles them to credit, in the various senses of the word [1].

The SC of a society includes the institutions, the relationships, the attitudes and values that that determine the quality and quantity of interactions among people and contribute to economic and social development. The notion that social relations, networks, norms, and values matter in the functioning and development of society has long been present in the economics, sociology, anthropology, and political science literature [2].

If we look beyond the role of the state, there remain at least two additional sources of SC. The first is religion. The second source of SC in developing countries is globalization. Globalization has been the bearer not just of capital but of ideas and culture as well [3]. It is not just investment bankers who can take advantage of the global communications and information revolution; activists of all sorts from environmentalists to labor organizers can now operate transnationally to a much greater extent than before [4].

In [5] the authors propose an approach of conversion a Z-number into a fuzzy number. The advantage of the given method is apparent by its low computational difficulty in its application. From informative structure point of view conversion of Z-numbers into classical fuzzy numbers [6-7] results in loss of original information.

In [8] some limitations and the disadvantages of Moore’s interval arithmetic are illustrated:

-the excess width effect;

-the dependency problem;

-difficulties in solution of simple interval equations;

-problems with the of right

-hand sides of the interval equations;

-senseless solutions and introduction of negative entropy into the system.

Moore arithmetic [9-11] can be substituted by multidimensional RDM interval arithmetic. Advantages of RDM arithmetic over the interval arithmetic are stated below [8]:

-complicated problems can be solved, due to the possibility of transforming equations;

-almost all laws of the of crisp numbers holds for RDM arithmetic;

-RDM arithmetic provides complete, multidimensional problem solutions from which various simplified representations such as cardinality distribution, a span of a solution (Moore’s solution) or a center of gravity can be derived.

The Multidimensional RDM arithmetic theory was developed by A.Piegat [12-15].

From the previous study of Z-number, the information of Z-number is given directly by the domain experts subjectively. In [16], a method of generating Z-number based on OWA weight using maximum entropy is initially proposed to guide the determination of Z-number to reduce the subjectivity of the experts. In addition, the membership function should be related to decision attitude of the expert. The proposed method [17] is more clear about the meaning of Z-number than previous study to deal with the reliability of Z-information.

Intuitionistic linguistic assessment of SC given in [18] is based on aggregation of SC indicators. In [19] PEO is made based on conversion of crisp data into Z-numbers. In this paper we developed a complex approach for evaluation of SC level referring to earlier works.

The paper is organized as follows. Section 2 illustrates intuitionistic linguistic assessment of SC using aggregation operator. In section 3 PEO is proposed where Z-numbers are converted into crisp values eventually. In section 4 controversial methodology for PEO is introduced, where crisp expert opinions are converted into Z-numbers.

Intuitionistic linguistic assessment of social capital level

In this section SC level in Azerbaijan is evaluated with the application of intuitionistic fuzzy tools.

With the purpose to extend SC indicators (SISC) developed by UN Basel Institute of Commons and Economics [20], the Healthcare and Corruption indicators are also added to the existing list:

1. Social climate (psychological environment, social conditions) is characteristically outlined as the concepts of a social atmosphere that is bound to be shared by the members of the society [21] – **SC**
2. The interpersonal trust among the people – to expect that someone is righteous and trustworthy and will not mistreat you – **TR**
3. Interest of the people to take individual strict measures with a view to fund public goods such as: safety, healthcare, education, environmental challenges, infrastructure, social allowances, media, culture – **PG**
4. Interest of the people to be involved in paying more taxes and making contributions to fund public goods such as: safety, healthcare, education, environmental challenges, infrastructure, social allowances, media, culture [20] – **PT**
5. Willingness of the people for adding up to national and regional level investments such as: purchasing of cooperative shares, acquisition of national and local stocks, purchasing of SME shares (small and medium enterprises), foundation of private or family business – **IE**
6. Helpfulness of the people – the character of offering necessary help, and (2) friendliness evinced by warm and gracious attitude [22] – **HE**
7. Friendliness of the people – The character of a person to be kind, and having affection for people – **FR**
8. Hospitality of the people – welcoming, entertaining, and generous treatment of guests, visitors or strangers - **HO**
9. Healthcare - Health care is the entire societal commitment, organized or not, either private or public, that aims to maintain, provide, fund, and improve health – **HL** [22]
10. Corruption is a severe crime that deflowers the social and economic growth and destroys the constitution of contemporary societies [23] – **CO**

The algorithm for estimation of sub-indices has been developed comprising the following steps:

Step 1. An intuitionistic linguistic number (ILN) A in X is defined [24] as following:

$$A = \{\langle x[h_{\theta(x)}, (\mu_A(x), \nu_A(x))]\rangle | x \in X\} \quad 1)$$

Here, $h_{\theta(x)} \in S$ and $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and non-membership degree of the element x related to linguistic index , $h_{\theta(x)}$ respectively.

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad 2)$$

for all x ∈ X. For each ILN A in X, if

$$\pi_{A(x)} = 1 - \mu_A(x) - \nu_A(x), \forall\, x \in X \quad 3)$$

then $\pi_{A(x)}$ is called the indeterminacy degree or hesitation degree of x of linguistic index $h_{\theta(x)}$.

Step 2. For computational convenience, let $S = \{s_\alpha | \alpha = 0, 1, \dots, l-1\}$ be a finite and totally ordered discrete term set, where l is the odd value and s_α represents a possible value for a linguistic variable. For example, when l = 7, a set S could be given as follows:

$S = s_0, s_1, s_2, s_3, s_4, s_5, s_6 =$ {very poor, poor, slightly poor, fair, slightly good, good, very good}

In this paper the set of interest is given as:

$$S = \{S_1 - Low, S_2 - Middle, S_3 - High\}$$

Step 3. Normalized indicators are converted into intuitionistic fuzzy numbers using the intuitionistic fuzzy triangular functions iftrif [25].

A definite intuitionistic fuzzy triangular membership and non-membership function of A is introduced as following:

$$\mu_A(x) = \begin{cases} 0 & ; \\ \left(\frac{x-a}{b-a}\right) - \epsilon; & \\ \left(\frac{c-x}{c-b}\right) - \epsilon; & \\ 0 & ; \end{cases} \qquad \nu_A(x)$$

$$= \begin{cases} 1 - \epsilon & ; \quad x \leq a \\ 1 - \left(\frac{x-a}{b-a}\right); & a < x \leq b \\ 1 - \left(\frac{c-x}{c-b}\right); & b \leq x < c \\ 1 - \epsilon & ; \quad x \geq c \end{cases} \quad 4)$$

Step 4. Weights of indicators are estimated as the weights proposed by decision makers. This concept is a more effective way to deal with vagueness of DMs, which may not be able to accurately express their satisfaction (or membership) degrees for indicators, due to that (1) the decision-makers (DM) have not precise or sufficient information about the problem; (2) the DMs are unable to discriminate explicitly the superiority of an indicator to others [26].

Let $D_k = [\mu_k, \nu_k, \pi_k]$ be an intuitionistic fuzzy number for rating of k-th decision maker. Then the weight of k-th decision maker can be obtained as:

$$\lambda_k = \frac{(\mu_k + \pi_k(\frac{\mu_k}{\nu_k}))}{\sum_{k=1}^{l}(\mu_k + \pi_k(\frac{\mu_k}{\nu_k}))} \quad 5)$$

And $\sum_{k=1}^{l} \lambda_k = 1.$

Step 5. The values of the sub-indices are estimated according to the following intuitionistic linguistic weighted average (ILWA) formula:

$$\boldsymbol{ILWA} = \langle S_{\sum_{k=1}^{t} \lambda_k \theta\left(a_{ij}^k\right)}, (1 - \prod_{k=1}^{t}(1 - \mu\left(a_{ij}^k\right))^{\lambda_k}, \prod_{k=1}^{t}(\nu\left(a_{ij}^k\right))^{\lambda_k}\rangle \quad 6)$$

After conversion of expert opinions into intuitionistic fuzzy numbers the following data (Table 1) is obtained for SC indicators:

Table 1. Intuitionistic fuzzy SC indicators

Indicators		SC	TR	PG	PT	IE	HE	FR	HO	HL	CO
Crisp values		6	6	5	5	7	9	8	9	6	7.5
Parameters of IFNs	μ	0.28	0.28	0.76	0.76	0.32	0.46	0.78	0.46	0.28	0.55
	ν	0.68	0.68	0.16	0.16	0.15	0.49	0.13	0.49	0.68	0.38
	π	0.03	0.03	0.09	0.09	0.53	0.05	0.09	0.05	0.03	0.07
Weights of criteria	λ	0.04	0.04	0.15	0.15	0.19	0.07	0.17	0.07	0.04	0.08

In computational process terms with following intervals were used: Low [1-3.3] Middle [3.0- 6.6] High [6.3-10].

$$\langle S_{2.58}(0.61, 0.22)\rangle$$

As seen from result the quality of SC is high. In future, this value of the quality of SC will be applicable in computation processes.

PREFERENCE OF EXPERT OPINIONS I

In the first methodology SC indicators (Table 1) given in Z-numbers as expert opinions will be fuzzified, following the normalization average graded representation technique and priority the expert opinion is selected out of three options.

3.1. Fuzzification of expert opinions data of social capital

The values of these indicators of Azerbaijan's SC are calculated on a 10 - point scale, taken from the report of the Basel Institute [20], which will be represented by fuzzy triangular numbers:

$$\tilde{6}, \tilde{6}, \tilde{5}, \tilde{5}, \tilde{7}, \tilde{9}, \tilde{8}, \tilde{9}, \tilde{6}, \tilde{7}$$

For example: $\tilde{6} = (5,6,7)$.

Let us set the opinions of three experts $\{s_1\}, \{s_2\}, \{s_3\}$ with fuzzy numbers $\widetilde{0.1}$, $\widetilde{0.4}$, $\widetilde{0.5}$, meaning low, close to middle, average levels of SC with high confidence in their assumptions, described by a fuzzy number $\tilde{1}$, that is, with $\widetilde{VH}(0.9, 1.0, 1.0)$.

The description of the set of states by means of Z-numbers is as follows:

Low (L)- $s_1: \tilde{Z}(\tilde{P}(s_1), \widetilde{B_1}) = \tilde{Z}\ (low, very\ high)$

Close to middle (CTM) - $s_2: \tilde{Z}(\tilde{P}(s_2), \widetilde{B_1}) = \tilde{Z}\ (close\ to\ middle, very\ high)$

Middle (M) - $s_3: \tilde{Z}(\tilde{P}(s_3), \widetilde{B_1}) = \tilde{Z}\ (middle, very\ high)$

Based on the above given conditions linguistic evolutions of the 10 criteria are given in Table 2.

Table 2. Expert opinions in linguistic terms

№	$\{s_1\}$ $((L),\widetilde{VH})$	$\{s_2\}$ $((CTM),\widetilde{VH})$	$\{s_3\}$ $((M),\widetilde{VH})$
1	((5,6,7),(H))	((5,6,7),(M))	((5,6,7),(L))
2	((5,6,7),(H))	((5,6,7),(M))	((5,6,7),(L))
3	((4,5,6),(H))	((4,5,6),(M))	((4,5,6),(L))
4	((4,5,6),(H))	((4,5,6),(M))	((4,5,6),(L))
5	((6,7,8),(H))	((6,7,8),(M))	((6,7,8),(L))
6	((8,9,10),(H))	((8,9,10),(M))	((8,9,10),(L))
7	((7,8,9),(H))	((7,8,9),(M))	((7,8,9),(L))
8	((8,9,10),(H))	((8,9,10),(M))	((8,9,10),(L))
9	((5,6,7),(H))	((5,6,7),(M))	((5,6,7),(L))
10	((6,7,8),(H))	((6,7,8),(M))	((6,7,8),(L))

Then we go to the decision – matrix with crisp values. Here *L, M, H* - fuzzy linguistic triangle variable in the interval [0, 1]. For example, *L* – Low (0, 0.165, 0.33), *M* – Middle (0.33, 0.495, 0.667), *H* – High (0.667, 0.83, 1.00).

Z –numbers in numerical values are described as follows:

Table 3. Fuzzification of expert opinion values

№	$\{s_1\}$ $((0,0.1,0.2),\widetilde{VH})$	$\{s_2\}$ $((0.3,0.4,0.5),\widetilde{VH})$	$\{s_3\}$ $((0.4,0.5,0.6),\widetilde{VH})$
1	((5,6,7),(0.667,0.83,1.00))	((5,6,7),(0.33,0.495,0.667))	((5,6,7),(0.0,0.165,0.33))
2	((5,6,7),(0.667,0.83,1.00))	((5,6,7),(0.33,0.495,0.667))	((5,6,7),(0.0,0.165,0.33))
3	((4,5,6),(0.667,0.83,1.00))	((4,5,6),(0.33,0.495,0.667))	((4,5,6),(0.0,0.165,0.33))
4	((4,5,6),(0.667,0.83,1.00))	((4,5,6),(0.33,0.495,0.667))	((4,5,6),(0.0,0.165,0.33))
5	((6,7,8),(0.667,0.83,1.00))	((6,7,8),(0.33,0.495,0.667))	((6,7,8),(0.0,0.165,0.33))
6	((8,9,10),(0.667,0.83,1.00))	((8,9,10),(0.33,0.495,0.667))	((8,9,10),(0.0,0.165,0.33))
7	((7,8,9),(0.667,0.83,1.00))	((7,8,9),(0.33,0.495,0.667))	((7,8,9),(0.0,0.165,0.33))
8	((8,9,10),(0.667,0.83,1.00))	((8,9,10),(0.33,0.495,0.667))	((8,9,10),(0.0,0.165,0.33))
9	((5,6,7),(0.667,0.83,1.00))	((5,6,7),(0.33,0.495,0.667))	((5,6,7),(0.0,0.165,0.33))
10	((6,7,8),(0.667,0.83,1.00))	((6,7,8),(0.33,0.495,0.667))	((6,7,8),(0.0,0.165,0.33))

For simplicity of the calculation procedure, we will normalize fuzzified expert opinion values in the following way:

$$x_n = \frac{x - x_{min}}{x_{max} - x_{min}} \tag{7}$$

The normalized value $x_n \in [0,1]$:
For example: if x = 5, then $x_n = \frac{5-1}{10-1} = 0,444$

3.2. Obtaining the priorities of expert opinions

In this section there is a necessity to introduce some definitions to cover the solution of the problem.

Definition 1 (Z-number). Let A, B are the two triangle numbers. $Z = (\tilde{A}, \tilde{B})$ - an ordered pair provided that $\tilde{A}$ is a constraint on the values of some fuzzy variable in X, and $\tilde{B}$ is an estimate of the confidence that X is in $\tilde{A}$. The initiation and use of the new concept of Z-numbers made it possible to solve decision-making problems under conditions of high-order [27].

Definition 2. Let $\tilde{A}(a_{1,}a_{2,}, a_{3,})$, $\tilde{B}(b_{1,}b_{2,}, b_{3,})$, are the two triangle fuzzy numbers. The average graded general representation [16] of these numbers are respectively:

$$P(\tilde{A}) = \frac{1}{6}(a_1 + 4a_2 + a_3)$$
$$P(\tilde{B}) = \frac{1}{6}(b_1 + 4b_2 + b_3) \tag{8}$$

Definition 3. The operator of multiplication [16,28] of triangular fuzzy numbers $\tilde{A}, \tilde{B}$ is calculated by the following formula:

$$P(\tilde{A}\otimes\tilde{B}) = P(\tilde{A}) * P(\tilde{B}) = \frac{1}{36}(a_1 + 4a_2 + a_3) * (b_1 + 4b_2 + b_3) \qquad (9)$$

Definition 4. Weight priority of each alternative [16,28] can be calculated as:

$$priority = \sum w(Z_a) * w(Z_j) \qquad (10)$$

where Z_a- weight criteria, Z_j- the value of each criterion.
Using the multiplication operator formula, we obtain a matrix of crisp values:
For example: $Z(s_3) = P(0.4,0.5,0,6)\otimes P(0.9,1.0,1.0) = \frac{1}{36}(3 * 5.9) = 0.492$

$$\begin{pmatrix} 0.098 & 0.393 & 0.492 \\ 0.461 & 0.245 & 0.092 \\ 0.461 & 0.245 & 0.092 \\ 0.369 & 0.220 & 0.073 \\ 0.369 & 0.220 & 0.073 \\ 0.553 & 0.331 & 0.110 \\ 0.738 & 0.441 & 0.147 \\ 0.646 & 0.386 & 0.128 \\ 0.738 & 0.441 & 0.147 \\ 0.461 & 0.245 & 0.092 \\ 0.584 & 0349 & 0.116 \end{pmatrix}$$

Taking into account the previously calculated weights of attributes by the $IFS\ (\lambda_i, i - 1, .., 10)$ method we obtain a new extended matrix [16]:

Table 4. Decision matrix in crisp values and criteria weights

$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	λ_i
0.461	0.245	0.092	0.04
0.461	0.245	0.092	0.04
0.369	0.220	0.073	0.15
0.369	0.220	0.073	0.15
0.553	0.331	0.110	0.19
0.738	0.441	0.147	0.07
0.646	0.386	0.128	0.17
0.738	0.441	0.147	0.07
0.461	0.245	0.092	0.04
0.584	0.349	0.116	0.08

3.3. Assessment of priorities of expert opinions: Next, the priority matrix is calculated using the formula:

$$priority = \sum w(Z_\lambda) * w(Z_k) \tag{11}$$

where Z_λ- weight criteria, and Z_k- the value of each criterion.
Next, we obtain a new matrix taking into account λ_i.

$$\begin{pmatrix} 0.0980 & 0.3930 & 0.4920 \\ 0.0184 & 0.0098 & 0.0037 \\ 0.0184 & 0.0098 & 0.0037 \\ 0.0550 & 0.0073 & 0.0109 \\ 0.0550 & 0.0073 & 0.0109 \\ 0.1051 & 0.0629 & 0.0209 \\ 0.0517 & 0.0309 & 0.0103 \\ 0.1098 & 0.0656 & 0.0218 \\ 0.0517 & 0.0309 & 0.0103 \\ 0.0184 & 0.0098 & 0.0037 \\ 0.0467 & 00279 & 0.0093 \end{pmatrix}$$

Applying formula (3) the following results are obtained:

$$priority(\exp 1) = 0.0519$$
$$priority(\exp 2) = 0.1695$$
$$priority(\exp 3) = 0.0521$$

The maximum value of the priority of opinions turned out to be the value of II Expert. This means that the state of SC is closer to the value of the existing number $(\widetilde{0.4}, \widetilde{VH})$.

PREFERENCE OF EXPERT OPINIONS II

The second proposed method of the complex fuzzy evaluation of SC is based on our previous work [27]. The core principle of this approach is the conversion of crisp data into Z-numbers grounded on modified method of generating Z-numbers proposed by Ye Tian and B. Kang [17].

4.1 Estimation of membership function values of generalized z-numbers

In this section to carry out PEO we consider SC indicators of Azerbaijan put forward by two experts. The expert opinions are stated in zero to ten scale [0-10] accordingly, where 0 – very bad and 10 – very high, which are represented in Table 5.

Table 5. Expert opinion data in crisp values

	SC	TR	PG	PT	IE	HE	FR	HO	HL	CO
Expert. Opinion 1	6	6	5	5	7	9	8	9	6	7.5
Expert. Opinion 2	9	9	5	6	6	10	10	10	7	9

In the utilization of expert data, three variants of weight distributions are possible to take into account:

normative - W^1[0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1],
intuitionistic - W^2[0.04,0.04,0.15,0.15,0.19,0.07,0.17,0.07,0.04,0.08] is referred to section 2, [Table 1],
optimistic - W^3[1,0,0,0,0,0,0,0,0,0].

The ORNESS measurement value that reflects numerically each expert's attitude (W), that was developed by Yager and Kacprzyk, [29] is formulated as following:

$$ORNESS(W) = \alpha = \frac{\sum_{i=1}^{n}(n-i) * w_i}{n-1} \qquad (12)$$

where w_i (i=1,2,..,10) are elements of the vector W^i (i=1,2,3).
ORNESS for normative distribution α_1, is computed as:

$$\alpha_1 = \frac{1}{9}\sum_{i=1}^{10}(n-i) * w_i = \frac{1}{9} * \frac{1}{10}\sum_{i=1}^{10}(10-i) = \frac{45}{90} = 0.5$$

In the same way α_2, α_3 are calculated:

$$\alpha_2 = 0.505, \qquad \alpha_3 = 1$$

Applying the maximum entropy formula, the probability distribution vector of the alternatives employed by the expert can be calculated, thus, taking into account α_i $(i = 1,2,3)$ the following optimization problem is executed:

$$\max H(W) = -\sum_{i=1}^{n} w_i * \ln(w_i)$$

$$\text{s.t} \quad \begin{cases} \alpha = \frac{\sum_{i=1}^{n}(n-i)*w_i}{n-1} \\ \sum_{i=1}^{n} w_i = 1 \\ 0 \le w_i \le 1 \end{cases} \qquad (13)$$

Next, the probability for the constraint part of A is defined as:

$$p_X(x_i) = w_{n-i+1} \qquad (14)$$

Further, with the reference to papers [5,6] the results are below:

P_n = [0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1]
P_i = [0.04,0.04,0.15,0.15,0.19,0.07,0.17,0.07,0.04,0.08],
P_o = [0,0,0,0,0,0,0,0,0,1]

Expert opinions on SC indicators introduced by expert I and II presented in crisp values are fuzzified, applying fuzzy triangle membership function (a = 6, b = 5.5, c = 10 is pre-established), that is illustrated in Table 6.

Table 6. Membership degree values regarding expert opinions on SC indicators

	SC	TR	PG	PT	IE	HE	FR	HO	HL	CO
I Expert opinion	0.89	0.89	0.89	0.89	0.67	0.22	0.44	0.22	0.89	0.55
II Expert opinion	0.67	0.67	0.89	0.89	0.89	0.00	0.00	0.00	0.67	0.67

Using data from Table 2, the constraint part (A_i) of Z-number for normative attitude is computed as:

$$A_1 = \frac{\mu_{A_1}}{x} = \frac{0.89}{6} + \frac{0.89}{6} + \frac{0.89}{5} + \frac{0.89}{5} + \frac{0.67}{7} + \frac{0.22}{9} + \frac{0.44}{8} + \frac{0.22}{9} + \frac{0.89}{6} + \frac{0.55}{7.5}$$

$$A_2 = \frac{\mu_{A_2}}{x} = \frac{0.67}{9} + \frac{0.67}{9} + \frac{0.89}{5} + \frac{0.89}{5} + \frac{0.89}{5} + \frac{0.00}{10} + \frac{0.00}{10} + \frac{0.00}{10} + \frac{0.67}{7} + \frac{0.67}{9}$$

Fuzzy probabilities A_1, A_2 of the argument X for the constraint part are estimated as:

$$\frac{P_x(x_i) * \mu_{A_1}(x_i)}{x_i} = \frac{0.89 * 0.1}{6} + \frac{0.89 * 0.1}{6} + \frac{0.89 * 0.1}{5} + \frac{0.89 * 0.1}{5} + \frac{0.67 * 0.1}{7} + \frac{0.22 * 0.1}{9} + \frac{0.44 * 0.1}{8} + \frac{0.22 * 0.1}{9} + \frac{0.89 * 0.1}{6} + \frac{0.55 * 0.1}{7.5}$$

$$\frac{P_x(x_i) * \mu_{A_2}(x_i)}{x_i} = \frac{0.67 * 01}{9} + \frac{0.67 * 0.1}{9} + \frac{0.89 * 0.1}{5} + \frac{0.89 * 0.1}{5} + \frac{0.89 * 0.1}{5} + \frac{0.00 * 0.1}{10} + \frac{0.00 * 0.1}{10} + \frac{0.00 * 0.1}{10} + \frac{0.67 * 0.1}{7} + \frac{0.67 * 0.1}{9}$$

4.2 Assessment of the reliability parts of z-numbers

The quantity of b of the reliability part (B) is obtained on the basis of calculation of $P(A)$, formulated as below:

$$b = p(A) = \int_R \mu_A(x) p_X(x) dx, b \in B \qquad (15)$$

where, R is the domain of real numbers.

The values of reliability part of B of Z-numbers for expert I and II are:

$$b_1^1 = P(A_1^1) = 0.655$$
$$b_2^1 = P(A_2^1) = 0.651$$
$$b_3^1 = P(A_3^1) = 0.55$$
$$b_1^2 = P(A_1^2) = 0.535$$
$$b_2^2 = P(A_2^2) = 0.565$$
$$b_3^2 = P(A_3^2) = 0.67$$

where, $i = 1,2,3$ are weight distributions, $j = 1,2$ are experts.

The values of fuzzy membership function are to be accepted equal to the values of *ORNESS*:

$$\mu_B(x) = \mu_{p_A}(p_A) = k\alpha_1$$
$$\mu_{B_1} = \mu_{PA_1(P_{A_1})} = \alpha_1 = 0.5$$
$$\mu_{B_2} = \mu_{PA_2(P_{A_2})} = \alpha_2 = 0.505$$
$$\mu_{B_3} = \mu_{PA_3(P_{A_3})} = \alpha_3 = 1$$

Next, the distances between the estimated probability distributions and the degrees of similarity and credibility have to be computed that allows to establish the most reliable version of expert opinions.

The Hellinger distance formula is applied in order to estimate the distances between probability distributions:

$$D_H(P,Q) = \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2} \qquad (16)$$

All feasible distances among distributions are also computed and the obtained results are shown below:

$D_H(P_1, P_2) = 0.185,$ $\quad D_H(P_1, P_3) = 0.805,$ $\quad D_H(P_2, P_3) = 0.843$

Components of similarly matrix between the probability distributions are calculated as:

$$Sim(p_i, p_j) = 1 - D_H(p_i, p_j) \qquad (17)$$

The estimated values are:

$Sim\ P_{12} = 0.815,$ $\quad Sim\ P_{13} = 0.195,$ $\quad Sim\ P_{23} = 0.157$

The similarity measure matrix (SMM) is constructed collecting the elements obtained above as shown below:

$$SMM = \begin{pmatrix} 1.00 & 0.815 & 0.195 \\ 0.815 & 1.00 & 0.157 \\ 0.195 & 0.157 & 1.00 \end{pmatrix}$$

The degree of support $Sup\ (P_j)$, is defined by the following expression :

$$Sup(P_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sim(P_i, P_j) \qquad (18)$$

As a result:

$$Sup(P_1) = 0.815 + 0.195 = 1.01$$
$$Sup(P_2) = 0.815 + 0.157 = 0.972$$
$$Sup(P_3) = 0.195 + 0.157 = 0.352$$

$$\sum_{i=1}^{3} Sup(P_i) = 2.334$$

Creditability degrees for probability distributions are assessed by the formula:

$$crd_j = \frac{Sup(P_j)}{\sum_{j=1}^{n} Sup(P_j)}\ , \qquad (i = 1,2,3) \qquad (19)$$

That can be interpreted as weight and can be employed as a discount coefficient to obtain the membership degree values of the reliability part B. The results appear as followings:

$$crd_1 = \frac{1,01}{2,334} = 0,433$$
$$crd_2 = \frac{0,972}{2,334} = 0,416$$
$$crd_3 = \frac{0,352}{2,334} = 0,151$$

The membership degree value of reliability part (B) of Z-number is computed as:

$$\mu_B^j(x) = \frac{k * \alpha_j * crd_j}{max(k * \alpha_j * crd_j)} \qquad j = (\overline{1,,n}) \qquad (20)$$

In this method, in favor of simplicity, the relation coefficient k among orness measure and membership function of B is pre-established as: $k = 1$

$$\alpha_1 = 0.5, \quad \alpha_2 = 0.505, \quad \alpha_3 = 1$$
$$k * \alpha_1 * crd_1 = 0.439 * 0.5 = 0.2165$$
$$k * \alpha_2 * crd_2 = 0.505 * 0.416 = 0.2101$$

$$k * \alpha_3 * crd_3 = 1 * 0.151 = 0.151$$

$$\mu_{B_1}(x) = \frac{0.5 * 0.433}{0.2165} = 1$$

$$\mu_{B_2}(x) = \frac{0.505 * 0.416}{0.2165} = 0.9703$$

$$\mu_{B_3}(x) = \frac{0.151 * 1}{0.2165} = 0.692$$

Lastly, the Z-numbers have been generated comprehensively and all factors included. In the analyzed example his makes possible to select the most reliable option that has the highest value of the B part of the Z-number, which matches the normative option. The acquired values of the Z-numbers for the normative attitude for both experts are:

$$B_1^1 = 0.5/0.655 \qquad B_1^2 = 0.5/0.535$$

Eventually, it is obvious that membership degrees of Z-numbers for both experts' opinions are equal for the normative option. Thus, the opinion of the I expert is preferred due to its highest reliability value.

Conclusion

In this paper, complex evaluation of SC is introduced. First, intuitionistic linguistic aggregation is employed to evaluate SC. Then we made PEO with the application of the theory and instruments of Z-numbers. For this purpose, we used two controversial methods. In the first approach expert opinions on SC is selected based on their priority values. For this purpose, initially the expert opinion values presented in Z-numbers are fuzzified, then converted in crisp values. Finally, the priorities of expert opinions are obtained. Based on the highest priority value the expert evaluation on SC representing actual level in the country is selected. Conversion of crisp data into Z-numbers is suitable to make PEO, on that account within the second approach we applied Kang's modified generalized Z-number method based on ordered weighted average and maximum entropy. Taking into consideration the reliability part of Z-numbers results in more reasonable solution that contemplates human judgements, and is an advantage over the traditional fuzzy methods in making PEO. At last intuitionistic linguistic aggregation is employed to evaluate SC level. The proposed work can be useful in solution of relevant socio-economic problems when conversion of Z-numbers in crisp values or vice-versa is a necessity.

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EVALUATION OF MICROCREDIT BORROWERS USING SOME METHODS OF SCORING ANALYSIS AND FUZZY INFERENCE

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A comprehensive analysis of the applicant's solvency for obtaining a microcredit precedes the conclusion of a loan agreement with him. This allows to determine the risk factors associated with the possibility of non-repayment of a bank loan in due time, and, on the contrary, to assess the likelihood of timely repayment of the loan. Therefore, the assessment of the client's creditworthiness is an integral part of the work of commercial banks and microfinance organizations to determine the possibility of issuing microloans to one or another applicant. The paper proposes a balanced approach to the multi-criteria assessment of the solvency of individuals, based, among other things, on a fuzzy analysis of their solvency indicators. The developed fuzzy inference system in combination with statistical methods for assessing solvency, can serve as an analytical core for a credit decision support system. Based on the example of ten hypothetical alternative borrowers, characterized by their current indicators, the corresponding assessments of their solvency were made, including scoring, Pareto method, Bord method and using a fuzzy inference system. Such a combined approach is distinguished by the ability to reliably identify a group of individuals with high credit discipline and the characteristics of those in relation to whom credit decisions are classified as high risk.

Keywords: microcredit; solvency indicator; Pareto method; Bord method; fuzzy inference.

PACS Number(s): PACS Number 1; PACS Number 2; PACS Number 3 (Enter a maximum of three PACS Numbers here) or

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1. INTRODUCTION

The scoring systems used by commercial banks and microcredit organizations use statistical methods of analysis that do not reflect the cause-effect relations between the objective and subjective characteristics of a potential borrower and his level of solvency at a certain date. To assessing the creditworthiness of individuals, in works [1, 2] there are proposed various fuzzy approaches based on the use of a fuzzy inference system (FIS) relative to the level of borrower's solvency. In [3, 4], we proposed a solution to the problem of assessing the microcredit potential borrowers using the fuzzy method of weighted maximin convolution of qualitative criteria for assessing creditworthiness, which provides for the fuzzification of applicants' personal data, i.e., their representation by appropriate fuzzy sets. Starting from fuzzy formalisms, in this article, as the main task, a multi-criteria assessment of microcredit potential borrowers is considered by using of the FIS.

2. PROBLEM DEFINITION

One of the modern tools for analyzing the borrower's solvency for a specific date is the so-called "scoring system", which, based on the applicant's data, promptly analyzes his credit history: financial transactions, delays, etc. In the case of microcredit, data about clients is taken as a basis directly from their applications, as well as, if necessary, from other available sources (for example, from social networks). One of the methods of scoring analysis is the scheme proposed in [5], according to which each potential microcredit borrower is assessed according to the following scheme.

- **Borrower's age (x_1),** which evaluated into the interval [0.1, 0.3]: the older the borrower, the more reliable he (she) is.
- **Borrower's sex (x_2),** which evaluated based on rule: 0 points for male, and 0.4 points for female, since the woman is considered as a more responsible borrower and less prone to risks and adventures.
- **Settledness (x_3),** which evaluated into the interval [0.042, 0.42]: the estimate directly depends on the time of permanent residence at the provided address.
- **Work risks (x_4),** which evaluated based on rule: 0 points – for borrowers working in hazardous productions, 0.16 points – for borrowers working under moderate risk to life, 0.55 points – for borrowers working in safe production.
- **Work in large companies (x_5):** 0.21 points are added to the overall grade of borrower, if he (she) works in a large enterprise.
- **Seniority (x_6),** which evaluated within the interval [0.059, 0.59]: the longer a borrower works, the more he (she) is reliable.
- **Assets (x_7):** 0.45 points are separately added, for example, for the presence of insurance, a deposit account and property.

To approve microcredit, the applicant must receive a final grade of at least 1.25 points, while the maximum score is 3.82.

Suppose that a commercial bank (or microfinance organization) considers applications from 10 individuals a_j (j = 1 ÷ 10) for the provision of microcredits to them. After verification of the applicants' personal data according to all the above criteria x_i (i =1 ÷ 7), preliminary data were obtained about each of them in the form of points awarded, which are summarized in Table 1 (see also [3, 4]). It is necessary to identify the best applicant for a microcredit among the applicants using the FIS. For this purpose, considering the preliminary data of the scoring analysis, it is necessary to formulate and build a cause-effect relation between the quality of the borrower's credit history and his level of solvency for a specific date in fuzzy set notation. In other words, it is necessary to build the FIS that, in the process of processing loan applications, would promptly ensure the aggregation of conclusions regarding the solvency of borrowers.

Table 1. Pre-testing results of the scoring.

Borrower	Evaluation criteria						
	x_1	x_2	x_3	x_4	x_5	x_6	x_7
a_1	0.161[1]	0.0	0.044	0.00	0.00	0.061	0.45
a_2	0.222	0.4	0.132	0.55	0.21	0.326	1.35
a_3	0.117	0.4	0.231	0.16	0.00	0.350	0.90
a_4	0.185	0.0	0.374	0.55	0.00	0.248	0.45
a_5	0.298	0.4	0.268	0.16	0.00	0.069	0.45
a_6	0.213	0.0	0.412	0.00	0.21	0.473	1.35
a_7	0.109	0.0	0.172	0.55	0.00	0.424	0.45
a_8	0.205	0.4	0.389	0.16	0.21	0.164	0.90
a_9	0.171	0.4	0.303	0.00	0.00	0.504	1.35
a_{10}	0.255	0.0	0.253	0.16	0.21	0.261	0.90

[1] Arbitrary personal data of microcredit applicants relative to the criteria for assessing their solvency.

3. MULTI-CRITERIA ASSESSMENT OF THE SOLVENCY OF MICRO-CREDIT BORROWERS USING THE FIS

Regarding the satisfactory solvency of a potential borrower, the management of a commercial bank adheres to certain rules, which are formulated in the form of the following judgments:

- ***e*$_1$:** “If the applicant's age is preferred and it has lived at the specified address for a long time, its work experience is preferable, and it also owns significant assets, then such borrower is satisfactory from the point of view of solvency”.
- ***e*$_2$:** “If, in addition to the above, the applicant works in an enterprise that is safe from the point of view of risk to life, then, in this case, applicant is more than satisfactory as a potential borrower”.
- ***e*$_3$:** “If, in addition to the requirements given in e1 and e2, the applicant is a woman working in a fairly large company, then such borrower is perfect (as fully complying with all credit requirements)”.
- ***e*$_4$:** “If everything stipulated in e3 takes place, except for the availability of assets, then the applicant is very satisfactory as a potential borrower”.
- ***e*$_5$:** “If the age of the applicant is not suitable and he lives at the specified address for a short time, however, applicant works in a large company and without much risk to its life, already has a solid work experience, but does not have certain assets, then applicant is still satisfactory as a potential borrower”.
- ***e*$_6$:** “If the applicant lives at the specified address for a short time, does not

have a solid permanent work experience and does not have any assets, then he (or she) is unsatisfactory as a potential borrower”.

The analysis of these statements allows us to reveal the full set of input and output characteristics of the future model in the form of the corresponding terms of the linguistic variables x_i (i = 1 ÷ 7) and the linguistic variable y = “*Satisfaction of the borrower's solvency*”, which are summarized in the following Table 2.

Table 2. Input and output characteristics of the fuzzy model.

Variable	Name	Terms	Universe
x_1	Age	F_1 = PREFERRED $\neg F_1$ = UNPREFERRED	[0.1, 0.3]
x_2	Sex	F_2 = DESIRED	{0, 0.4}
x_3	Settled way of life	F_3 = LONG CONTINUED $\neg F_3$ = SHORT-LIVED	[0.042, 0.42]
x_4	Work risks	F_4 = ACCEPTABLE	[0, 0.55]
x_5	Day-to-day work	F_5 = PREFERABLE	{0, 0.21}
x_6	Work experience	F_6 = LONG $\neg F_6$ = SHORT	[0.059, 0.59]
x_7	Availability of assets	F_7 = SIGNIFICANT $\neg F_7$ = INSIGNIFICANT	[0, 1.35]
y	Borrower satisfaction	UN = UNSATISFACTORY S = SATISFACTORY MS = MORE THAN SATISFACTORY VS = VERY SATISFACTORY P = PERFECT	{0, 0.1, ..., 1}

The evaluation concepts x_i (i = 1 ÷ 7) of the scoring analysis, according to which points are assign to potential borrowers, are considered as qualitative categories (criteria), and the numerical assessments of the solvency of alternative microcredit borrowers are the degrees of compliance with these criteria. Starting from this assumption, the set of alternatives (borrowers) is denoted as $A = \{a_1, a_2, \ldots, a_{10}\}$, and the set of criteria is denoted as $F = \{F_1, F_2, \ldots, F_m\}$, where, according to [6], each criterion can be described by the appropriate fuzzy subset of the universe A in the following form:

$$F_i = \frac{\mu_{F_i}(a_1)}{a_1} + \frac{\mu_{F_i}(a_2)}{a_2} + \cdots + \frac{\mu_{F_i}(a_{10})}{a_{10}},$$

As membership functions restoring this kind of fuzzy sets, following

Gaussian functions are chosen [6]:

$$\mu(u) = e^{-\frac{(u-u_i)^2}{\sigma_i^2}}, u \in [0, u_i],$$

where $i = 1 \div 7$; u_i is the maximum score provided for in the i-th scoring point; $\sigma_i^2 = \frac{1}{n}\sum_{k=1}^{n}(u_k - u_{ik})^2$ is the standard deviation. In particular, for the fuzzification of the qualitative evaluated criterion F_1 = PREFERRED (age), the Gaussian membership function with the density $\sigma_i^2 = 0.0138$ (n = 40) was chosen.

Thus, according to (1) and (2), the evaluative concepts, as criteria for assessing the solvency of borrowers, can be described in the form of the following corresponding fuzzy sets:

- PREFERRED (age): $F_1 = \frac{0.2475}{a_1} + \frac{0.6442}{a_2} + \frac{0.0889}{a_3} + \frac{0.3845}{a_4} + \frac{0.9997}{a_5} + \frac{0.5787}{a_6} + \frac{0.0716}{a_7} + \frac{0.5209}{a_8} + \frac{0.3004}{a_9} + \frac{0.8639}{a_{10}}$;
- DESIRED (sex): $F_2 = \frac{0.0081}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{0.0081}{a_4} + \frac{1}{a_5} + \frac{0.0081}{a_6} + \frac{0.0081}{a_7} + \frac{1}{a_8} + \frac{1}{a_9} + \frac{0.0081}{a_{10}}$;
- LONG CONTINUED (settled way of life): $F_3 = \frac{0.0564}{a_1} + \frac{0.1851}{a_2} + \frac{0.4837}{a_3} + \frac{0.9579}{a_4} + \frac{0.6251}{a_5} + \frac{0.9987}{a_6} + \frac{0.2863}{a_7} + \frac{0.9806}{a_8} + \frac{0.7570}{a_9} + \frac{0.5672}{a_{10}}$;
- ACCEPTABLE (work risks): $F_4 = \frac{0.0528}{a_1} + \frac{1}{a_2} + \frac{0.2279}{a_3} + \frac{1}{a_4} + \frac{0.6251}{a_5} + \frac{0.0528}{a_6} + \frac{1}{a_7} + \frac{0.2279}{a_8} + \frac{0.0528}{a_9} + \frac{0.2279}{a_{10}}$;
- PREFERABLE (day-to-day work): $F_5 = \frac{0.0555}{a_1} + \frac{1}{a_2} + \frac{0.0555}{a_3} + \frac{0.0555}{a_4} + \frac{0.0555}{a_5} + \frac{1}{a_6} + \frac{0.0555}{a_7} + \frac{1}{a_8} + \frac{0.0555}{a_9} + \frac{1}{a_{10}}$;
- LONG (work experience): $F_6 = \frac{0.0565}{a_1} + \frac{0.4888}{a_2} + \frac{0.5534}{a_3} + \frac{0.3008}{a_4} + \frac{0.0616}{a_5} + \frac{0.8688}{a_6} + \frac{0.7535}{a_7} + \frac{0.1551}{a_8} + \frac{0.9269}{a_9} + \frac{0.3290}{a_{10}}$;
- SIGNIFICANT (availability of assets): $F_7 = \frac{0.2742}{a_1} + \frac{1}{a_2} + \frac{0.7236}{a_3} + \frac{0.2742}{a_4} + \frac{0.2742}{a_5} + \frac{1}{a_6} + \frac{0.2742}{a_7} + \frac{0.7236}{a_8} + \frac{1}{a_9} + \frac{0.7236}{a_{10}}$.

To describe the output characteristics of the model the appropriate universe is chosen as discrete set $U = \{0, 0.1, 0.2, \ldots, 1\}$. Then, according to [7], $\forall u \in U$ the terms of the linguistic variable y from the right-hand sides of the rules $e_1 \div e_6$ can be described in the form of the following fuzzy sets with the corresponding membership functions:

- S = SATISFACTORY: $\mu_S(u) = u$;
- MS = MORE THAN SATISFACTORY: $\mu_{MS}(u) = \sqrt{u}$;
- VS = VERY SATISFACTORY: $\mu_{VS}(u) = u^2$;
- P = PERFECT: $\begin{cases} \mu_P(u) = 1, \text{if } u = 1, \\ \mu_P(u) = 0, \text{if } u < 1; \end{cases}$
- US = UNSATISFACTORY: $\mu_{US}(u) = 1 - u$.

Thus, taking into account the introduced designations, the rules $e_1 \div e_6$ can be represented in the form of the fuzzy implications:

- e_1: If x_1 is F_1 and x_3 is F_3 and x_6 is F_6 and x_7 and F_7, then y is S;
- e_2: If x_1 is F_1 and x_3 is F_3 and x_4 is F_4 and x_6 is F_6 and x_7 is F_7, then y is MS;
- e_3: If x_1 is F_1 and x_2 is F_2 and x_3 is F_3 and x_4 is F_4 and x_5 is F_5 and x_6 is F_6 and x_7 is F_7, then y is P;
- e_4: If x_1 is F_1 and x_2 is F_2 and x_3 is F_3 and x_4 is F_4 and x_5 is F_5 and x_6 is F_6, then y is VS;
- e_5: If x_1 is $\neg F_1$ and x_3 is $\neg F_3$ and x_4 is F_4 and x_5 is F_5 and x_6 is F_6 and x_7 is $\neg F_7$, then y is S;
- e_6: If x_3 is $\neg F_3$ and x_6 is $\neg F_6$ and x_7 is $\neg F_7$, then y is US.

Guided by the rule of intersection of fuzzy sets, for the left-hand sides of the rules $e_1 \div e_6$, we have, respectively:

- $\mu_{M1}(a) = \min\{\mu_{F1}(a), \mu_{F3}(a), \mu_{F6}(a), \mu_{F7}(a)\}$, $M_1 = \{0.0564/a_1; 0.1851/a_2; 0.0889/a_3; 0.2742/a_4; 0.0616/a_5; 0.5787/a_6; 0.0716/a_7; 0.1551/a_8; 0.3004/a_9; 0.3290/a_{10}\}$;
- $\mu_{M2}(a) = \min\{\mu_{F1}(a), \mu_{F3}(a), \mu_{F4}(a), \mu_{F6}(a), \mu_{F7}(a)\}$, $M_2 = \{0.0528/a_1; 0.1851/a_2; 0.0889/a_3; 0.2742/a_4; 0.0616/a_5; 0.0528/a_6; 0.0716/a_7; 0.1551/a_8; 0.0528/a_9; 0.2279/a_{10}\}$;
- $\mu_{M3}(a) = \min\{\mu_{A1}(a), \mu_{A2}(a), \mu_{A3}(a), \mu_{A4}(a), \mu_{A5}(a), \mu_{F6}(a), \mu_{F7}(a)\}$, $M_3 = \{0.0081/a_1; 0.1851/a_2; 0.0555/a_3; 0.0081/a_4; 0.0555/a_5; 0.0081/a_6; 0.0081/a_7; 0.1551/a_8; 0.0528/a_9; 0.0081/a_{10}\}$;
- $\mu_{M4}(a) = \min\{\mu_{A1}(a), \mu_{A2}(a), \mu_{A3}(a), \mu_{A4}(a), \mu_{A5}(a), \mu_{F6}(a)\}$, $M_4 = \{0.0081/a_1; 0.1851/a_2; 0.0555/a_3; 0.0081/a_4; 0.0555/a_5; 0.0081/a_6; 0.0081/a_7; 0.1551/a_8; 0.0528/a_9; 0.0081/a_{10}\}$;
- $\mu_{M5}(a) = \min\{1 - \mu_{A1}(a), 1 - \mu_{A3}(a), \mu_{A4}(a), \mu_{A5}(a), \mu_{F6}(a), 1 - \mu_{17}(a)\}$, $M_5 = \{0.0528/a_1; 0.0000/a_2; 0.0555/a_3; 0.0421/a_4; 0.0003/a_5; 0/a_6; 0.0555/a_7; 0.0194/a_8; 0/a_9; 0.1361/a_{10}\}$;
- $\mu_{M6}(a) = \min\{1 - \mu_{A3}(a), 1 - \mu_{A6}(a), 1 - \mu_{A7}(a)\}$, $M_6 = \{0.7258/a_1; 0/a_2; 0.2764/a_3; 0.0421/a_4; 0.3749/a_5; 0/a_6; 0.2465/a_7; 0.0194/a_8; 0/a_9; 0.2764/a_{10}\}$.

As a result, the rules $e_1 \div e_6$ are presented in a more compact form as follows:

- e_1: If x is M_1, then y is S;
- e_2: If x is M_2, then y is MS;
- e_3: If x is M_3, then y is P;
- e_4: If x is M_4, then y is VS;
- e_5: If x is M_5, then y is S;
- e_6: If x is M_6, then y is US.

Further, based on the Lukasiewicz’s implication

$$\mu(a,u) = \min\{1, 1 - \mu(a) + \mu(u)\},$$

the fuzzy relations are formed in the form of following corresponding matrices:

$R_1 =$

	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
0,0564	0,9436	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,1851	0,8149	0,9149	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0889	0,9111	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,2742	0,7258	0,8258	0,9258	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0616	0,9384	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,5787	0,4213	0,5213	0,6213	0,7213	0,8213	0,9213	1,0000	1,0000	1,0000	1,0000	1,0000
0.0716	0.9284	1.0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,1551	0,8449	0,9449	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,3004	0,6996	0,7996	0,8996	0,9996	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,3290	0,6710	0,7710	0,8710	0,9710	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000

,

$R_2 =$

	0	0,3162	0,4472	0,5477	0,6325	0,7071	0,7746	0,8367	0,8944	0,9487	1
0,0528	0,9472	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,1851	0,8149	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0889	0,9111	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,2742	0,7258	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0616	0,9384	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0528	0,9472	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0716	0,9284	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,1551	0,8449	1,0000	1,0000	1.0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0528	0,9472	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,2279	0,7721	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000

,

$R_3 =$

	0	0	0	0	0	0	0	0	0	0	1
0,0081	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	1,0000
0,1851	0,8149	0,8149	0,8149	0,8149	0,8149	0,8149	0,8149	0,8149	0,8149	0,8149	1,0000
0,0555	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	1,0000
0,1851	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	1,0000
0,0555	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	0,9445	1,0000
0,0081	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	1,0000
0,0081	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	1,0000
0,1551	0,8449	0,8449	0,8449	0,8449	0,8449	0,8449	0,8449	0,8449	0,8449	0,8449	1,0000
0,0528	0,9472	0,9472	0,9472	0,9472	0,9472	0,9472	0,9472	0,9472	0,9472	0,9472	1,0000
0,0081	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	0,9919	1,0000

,

$R_4 =$

	0	0,01	0,04	0,09	0,16	0,25	0,36	0,49	0,64	0,81	1
0,0081	0,9919	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,1851	0,8149	0,8249	0,8549	0,9049	0,9749	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0555	0,9445	0,9545	0,9845	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0081	0,9919	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0555	0,9445	0,9545	0,9845	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0081	0,9919	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0081	0,9919	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,1551	0,8449	0,8549	0,8849	0,9349	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0528	0,9472	0,9572	0,9872	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0081	0,9919	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000

,

$R_5 =$

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0,0528	0,9472	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0555	0,9445	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0421	0,9579	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0003	0,9997	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0555	0,9445	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0194	0,9806	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,1361	0,8639	0,9639	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000

,

$R_6 =$

	1,0	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1	0
0,7258	1,0000	1,0000	1,0000	0.9742	0.8742	0.7742	0.6742	0.5742	0.4742	0.3742	0.2742
0,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,2764	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0.9236	0.8236	0.7236
0,0421	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0.9579
0,3749	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0.9251	0.8251	0.7251	0.6251
0,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,2465	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0.9535	0.8535	0.7535
0,0194	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0.9806
0,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
0,2764	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0.9236	0.8236	0.7236

.

Intersection of these matrices ultimately gives a general functional solution in the form of the following matrix:

$R =$

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
a_1	0.9436	0.9919	0.9919	0.9742	0.8742	0.7742	0.6742	0.5742	0.4742	0.3742	0.2742
a_2	0.8149	0.8149	0.8149	0.8149	0.8149	0.8149	0.8149	0.8149	0.8149	0.8149	1.0000
a_3	0.9111	0.9445	0.9445	0.9445	0.9445	0.9445	0.9445	0.9445	0.9236	0.8236	0.7236
a_4	0.7258	0.8258	0.9258	0.9919	0.9919	0.9919	0.9919	0.9919	0.9919	0.9919	0.9579
a_5	0.9384	0.9445	0.9445	0.9445	0.9445	0.9445	0.9445	0.9251	0.8251	0.7251	0.6251
a_6	0.4213	0.5213	0.6213	0.7213	0.8213	0.9213	0.9919	0.9919	0.9919	0.9919	1.0000
a_7	0.9284	0.9919	0.9919	0.9919	0.9919	0.9919	0.9919	0.9919	0.9535	0.8535	0.7535
a_8	0.8449	0.8449	0.8449	0.8449	0.8449	0.8449	0.8449	0.8449	0.8449	0.8449	0.9806
a_9	0.6996	0.7996	0.8996	0.9472	0.9472	0.9472	0.9472	0.9472	0.9472	0.9472	1.0000
a_{10}	0.6710	0.7710	0.8710	0.9710	0.9919	0.9919	0.9919	0.9919	0.9236	0.8236	0.7236

According to [6, 7], a fuzzy conclusion relative to the solvency of the *j*-th potential borrower is reflected in the form of the fuzzy subset E_k of the universe *U* with the corresponding values of the membership function from the *j*-th row of the matrix *R*.

For numerical estimates of the borrower solvency defuzzification is applied. For example, for the fuzzy conclusion relative to the estimate of the 1[st] applicant solvency: E_8= {0.9436/0, 0.9919/0.1, 0.9919/0.2, 0.9742/0.3, 0.8742/0.4, 0.7742/0.5, 0.6742/0.6, 0.5742/0.7, 0.4742/0.8, 03742/0.9, 0.2742/1}, establishing the level-sets $E_{1\alpha}$ and calculating their corresponding cardinal numbers as $M(E_\alpha) = \frac{1}{n}\sum_{j=1}^{n} r_j$, we have:

- for $0 < \alpha < 0.2742$: $\Delta\alpha = 0.2742$, $E_{1\alpha} = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$, $M(E_{1\alpha}) = 0.50$;
- for $0.2742 < \alpha < 0.3742$: $\Delta\alpha = 0.1$, $E_{1\alpha} = \{0, 0.1, \ldots, 0.8, 0.9\}$, $M(E_{1\alpha}) = 0.45$;
- for $0.3742 < \alpha < 0.4742$: $\Delta\alpha = 0.1$, $E_{1\alpha} = \{0, 0.1, \ldots, 0.7, 0.8\}$, $M(E_{1\alpha}) = 0.40$;
- for $0.4742 < \alpha < 0.5742$: $\Delta\alpha = 0.1$, $E_{1\alpha} = \{0, 0.1, \ldots, 0.6, 0.7\}$, $M(E_{1\alpha}) = 0.35$;
- for $0.5742 < \alpha < 0.6742$: $\Delta\alpha = 0.1$, $E_{1\alpha} = \{0, 0.1, \ldots, 0.5, 0.6\}$, $M(E_{1\alpha}) = 0.30$;
- for $0.6742 < \alpha < 0.7742$: $\Delta\alpha = 0.1$, $E_{1\alpha} = \{0, 0.1, 0.2, \ldots, 0.5\}$, $M(E_{1\alpha}) = 0.25$;
- for $0.7742 < \alpha < 0.8742$: $\Delta\alpha = 0.1$, $E_{1\alpha} = \{0, 0.1, 0.2, 0.3, 0.4\}$, $M(E_{1\alpha}) = 0.20$;
- for $0.8742 < \alpha < 0.9436$: $\Delta\alpha = 0.0694$, $E_{1\alpha} = \{0, 0.1, 0.2, 0.3\}$, $M(E_{1\alpha}) = 0.15$;
- for $0.9436 < \alpha < 0.9742$: $\Delta\alpha = 0.0306$, $E_{1\alpha} = \{0.1, 0.2, 0.3\}$, $M(E_{1\alpha}) = 0.20$;
- for $0.9742 < \alpha < 0.9919$: $\Delta\alpha = 0.0177$, $E_{1\alpha} = \{0.1, 0.2\}$, $M(E_{1\alpha}) = 0.15$.

Then, according to [7], the numerical estimate is obtained in the following form:

$$F(E_1) = \frac{1}{\alpha_{\max}} \int_0^{\alpha_{\max}} M(E_{1\alpha})d\alpha = \frac{1}{0.9919} \int_0^{0.9919} M(E_{1\alpha})d\alpha = \frac{1}{0.9919}[0.2742 \cdot 0.50 + 0.1 \cdot 0.45 + 0.1 \cdot 0.40 + 0.1 \cdot 0.35 + 0.1 \cdot 0.30 + 0.1 \cdot 0.25 + 0.1 \cdot 0.20 + 0.0694 \cdot 0.15 + 0.0306 \cdot 0.20 + 0.0177 \cdot 0.15.$$

Numerical estimates of solvency for other potential borrowers are established in the same way: $F(E_2) = 0.5926$, $F(E_3) = 0.4826$, $F(E_4) = 0.5234$, $F(E_5) = 0.4645$, $F(E_6) = 0.5967$, $F(E_7) = 0.4823$, $F(E_8) = 0.5692$, $F(E_9) = 0.5485$, $F(E_{10}) = 0.5090$.

4. MULTI-CRITERIA ASSESSMENT OF THE SOLVENCY OF MICROCREDIT BORROWERS USING THE METHODS OF PARETO AND BORD

The Pareto method provides for the selection of the most solvent borrowers from applicants for a microcredit and, according to the total volume of their lending, an advances portfolio is formed [8]. At the 1st stage borrowers are ranked on the base of their data according with the system of criteria x_i ($i = 1 \div 7$). For the applicants under consideration (see Table 1), this ranking is summarized in Table 3.

Table 3. Ranking of alternative microcredit borrowers by solvency indicators

Order	Solvency indicators of comparative assessment of borrowers						
	x_1	x_2	x_3	x_4	x_5	x_6	x_7
1	a_5	a_5	a_6	a_4	a_2	a_9	a_9
2	a_{10}	a_2	a_8	a_7	a_8	a_6	a_6
3	a_2	a_8	a_4	a_2	a_{10}	a_7	a_2
4	a_6	a_9	a_9	a_8	a_6	a_3	a_3
5	a_8	a_3	a_5	a_5	a_4	a_2	a_{10}
6	a_4	a_{10}	a_{10}	a_{10}	a_7	a_{10}	a_8
7	a_9	a_6	a_3	a_3	a_5	a_4	a_7
8	a_1	a_4	a_7	a_6	a_3	a_8	a_4
9	a_3	a_1	a_2	a_9	a_9	a_5	a_5
10	a_7	a_7	a_1	a_1	a_1	a_1	a_1

At the next 2nd step, a comparative analysis of borrowers is carried out by establishing pairwise preferences according to the following principle: for segment “borrower a_k”, the sign “–” is set in the cell, where row x_i and column a_j intersect, because the value of the indicator x_i for the borrower a_k is less than for the borrower a_j, and at the intersection with the column a_r there is a sign “+”, because the value of the indicator x_i for the borrower a_k is greater than for the borrower a_r. If the indicators of borrowers are equal, then the sign “0” is fixed. According to the Pareto rule, if columns do not contain the sign “–”, then corresponding borrowers are preferred.

For example (see Table 4), for borrowers a_2 and a_8, column a_1 contains only signs “+”, which means that borrowers a_2 and a_8 are preferable to borrower a_1. Similar pairwise comparisons have shown that other borrowers are also preferred over a_1. Therefore, excluding the alternative a_1, the Pareto method performs pairwise comparisons of the remaining borrowers, which is easily simulated on a computer due to the triviality of the algorithm.

Table 4. Table of pairwise preferences of alternative borrowers

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
x_1	-	+	-	-	-	+	-	-	-
x_2	-	-	0	-	0	0	-	-	0
x_3	-	-	-	-	-	-	-	-	-
x_4	-	-	-	-	0	-	-	0	-
x_5	-	0	0	0	-	0	-	0	-
x_6	-	-	-	-	-	-	-	-	-
x_7	-	-	0	0	-	0	-	-	-
a_2	**a_1**	**a_3**	**a_4**	**a_5**	**a_6**	**a_7**	**a_8**	**a_9**	**a_{10}**
x_1	+	+	+	-	+	+	+	+	-
x_2	+	0	+	0	+	+	0	0	+
x_3	+	-	-	-	-	-	-	-	-
x_4	+	+	0	+	+	0	+	+	+
x_5	+	+	+	+	0	+	0	+	0
x_6	+	-	+	+	-	-	+	-	+
x_7	+	+	+	+	0	+	+	0	+
a_3	**a_1**	**a_2**	**a_4**	**a_5**	**a_6**	**a_7**	**a_8**	**a_9**	**a_{10}**
x_1	-	-	-	-	-	+	-	-	-
x_2	+	0	+	0	+	+	0	0	+
x_3	+	+	-	-	-	+	-	-	-
x_4	+	-	-	0	+	-	0	+	0
x_5	0	-	0	0	-	0	-	0	-
x_6	+	+	+	+	-	-	+	-	+
x_7	+	-	+	+	-	+	0	-	0
a_4	**a_1**	**a_2**	**a_3**	**a_5**	**a_6**	**a_7**	**a_8**	**a_9**	**a_{10}**
x_1	+	-	+	-	-	+	-	+	-
x_2	0	-	-	-	0	0	-	-	0
x_3	+	+	+	+	-	+	-	+	+
x_4	+	0	+	+	+	0	+	+	+
x_5	0	-	0	0	-	0	-	0	-
x_6	+	-	-	+	-	-	+	-	-

x_7	0	-	-	0	-	0	-	-	-
a_5	a_1	a_2	a_3	a_4	a_6	a_7	a_8	a_9	a_{10}
x_1	+	+	+	+	+	+	+	+	+
x_2	+	0	0	+	+	+	0	0	+
x_3	+	+	+	-	-	+	-	-	+
x_4	+	-	0	-	+	-	0	+	0
x_5	0	-	0	0	-	0	-	0	-
x_6	+	-	-	-	-	-	-	-	-
x_7	0	-	-	0	-	0	-	-	-
a_6	a_1	a_2	a_3	a_4	a_5	a_7	a_8	a_9	a_{10}
x_1	+	-	+	+	-	+	+	+	-
x_2	0	-	-	0	-	0	-	-	0
x_3	+	+	+	+	+	+	+	+	+
x_4	0	-	-	-	-	-	-	0	-
x_5	+	0	+	+	+	+	0	+	0
x_6	+	+	+	+	+	+	+	-	+
x_7	+	0	+	+	+	+	+	0	+
a_7	a_1	a_2	a_3	a_4	a_5	a_6	a_8	a_9	a_{10}
x_1	-	-	-	-	-	-	-	-	-
x_2	0	-	-	0	-	0	-	-	0
x_3	+	+	-	-	-	-	-	-	-
x_4	+	0	+	0	+	+	+	+	+
x_5	0	-	0	0	0	-	-	0	-
x_6	+	+	+	+	+	-	+	-	+
x_7	0	-	-	0	0	-	-	-	-
a_8	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_9	a_{10}
x_1	+	-	+	+	-	-	+	+	-
x_2	+	0	0	+	0	+	+	0	+
x_3	+	+	+	+	+	-	+	+	+
x_4	+	-	0	-	0	+	-	+	0
x_5	+	0	+	+	+	0	+	+	0
x_6	+	-	-	-	+	-	-	-	-
x_7	+	-	0	+	+	-	+	-	0
a_9	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_{10}
x_1	+	-	+	-	-	-	+	-	-
x_2	+	0	0	+	0	+	+	0	+
x_3	+	+	+	-	+	-	+	-	+
x_4	0	-	-	-	-	0	-	-	-
x_5	0	-	0	0	0	-	0	-	-
x_6	+	+	+	+	+	+	+	+	+
x_7	+	0	+	+	+	0	+	+	+
a_{10}	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
x_1	+	+	+	+	-	+	+	+	+
x_2	0	-	-	0	-	0	0	-	-
x_3	+	+	+	-	-	-	+	-	-
x_4	+	-	0	-	0	+	-	0	+
x_5	+	0	+	+	+	0	+	0	+
x_6	+	-	-	+	+	-	-	+	-
x_7	+	-	0	+	+	-	+	0	-

The Pareto method provides more credit decisions than it is necessary. Therefore, to complete the estimation process of the borrower solvency the Bord rule is used. According to this rule the alternative borrowers are ranked for each indicator on a ten-point system in order of descending with the appropriation of the corresponding ranks (see Table 5). The total rank is determined for each credit decision (see Table 6). Wherein, the borrower with the highest summarized rank is the best.

Table 5. Ranking potential borrowers using the Bord’s method

Rank	Criteria for comparative assessment of borrowers						
	x_1	x_2	x_3	x_4	x_5	x_6	x_7
10	a_5	a_5	a_6	a_4	a_2	a_9	a_9
9	a_{10}	a_2	a_8	a_7	a_8	a_6	a_6
8	a_2	a_8	a_4	a_2	a_{10}	a_7	a_2
7	a_6	a_9	a_9	a_8	a_6	a_3	a_3
6	a_8	a_3	a_5	a_5	a_4	a_2	a_{10}
5	a_4	a_{10}	a_{10}	a_{10}	a_7	a_{10}	a_8
4	a_9	a_6	a_3	a_3	a_5	a_4	a_7
3	a_1	a_4	a_7	a_6	a_3	a_8	a_4
2	a_3	a_1	a_2	a_9	a_9	a_5	a_5
1	a_7	a_7	a_1	a_1	a_1	a_1	a_1

Table 6. Total ranks of compared potential borrowers

Borrower	Criteria for comparative assessment of borrowers							Points total	Order
	x_1	x_2	x_3	x_4	x_5	x_6	x_7		
a_1	3	2	1	1	1	1	1	10	10
a_2	8	9	2	8	10	6	8	51	1
a_3	2	6	4	4	3	7	7	33	8
a_4	5	3	8	10	6	4	3	39	7
5	10	10	6	6	4	2	2	40	6
a_6	7	4	10	3	7	9	9	49	2
a_7	1	1	3	9	5	8	4	31	9
a_8	6	8	9	7	9	3	5	47	3
a_9	4	7	7	2	2	10	10	42	5
a_{10}	9	5	5	5	8	5	6	43	4

5. CONCLUSION

The FIS can also be useful for developing a balanced solution for the provision of microcredit. The results of assessing the current solvency of alternative borrowers a_j (j = 1 ÷ 10) in comparison with the results obtained by the Pareto and Bord methods are summarized in Table 7. The ranking of alternative borrowers obtained using arithmetic averaging and scoring analysis, completely coincide and differ insignificantly from the ordinal estimates obtained using the Pareto and Bord methods. Some difference is observed when comparing the statistical results with the ranking of borrowers obtained using the FIS, especially in terms of the swing estimate of the best borrower. The ordinal estimates obtained by the methods of scoring analysis, Pareto and Bord do not consider the specific weights of the criteria for assessing the borrower solvency, while the structure of logical rules (verbal model) implicitly considers the priority of some indicators of solvency over others.

Table 7. Results of assessments of the current solvency of microcredit borrowers

Borrower	Average		Scoring		Pareto rule	Bord method		FIS	
	Est.	Rank	Est.	Rank	Rank	Est.	Rank	Est.	Rank
a_1	0.1023	10	0.72	10	10	10	10	0.3542	10
a_2	0.4557	1	3.19	1	1	51	1	0.5926	2
a_3	0.3083	5	2.16	5	7	33	8	0.4826	7
a_4	0.2581	7	1.81	7	6	39	7	0.5234	5
a_5	0.2350	9	1.65	9	9	40	6	0.4645	9
a_6	0.3797	3	2.66	3	3	49	2	0.5967	1
a_7	0.2436	8	1.71	8	8	31	9	0.4823	8
a_8	0.3469	4	2.43	4	2	47	3	0.5692	3
a_9	0.3897	2	2.73	2	4	42	5	0.5485	4
a_{10}	0.2913	6	2.04	6	5	43	4	0.5090	6

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FUZZY EVALUATION OF AIR QUALITY OF AZERBAIJAN

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In this paper with the purpose to analyze the air quality in the country, an approach with the application of intuitionistic fuzzy linguistic tools is proposed. As known air quality is one of the main factors of environmental sustainability. Environmental sustainability Indicators are measured taking into account its different aspects. Intuitionistic Fuzzy Set (IFS) is a very powerful tool for processing uncertain information. IFS are characterized by the degree of membership and the degree of non-membership. Similar to IFS, linguistic intuitionistic fuzzy set (LIFS) is characterized by a linguistic membership degree and a linguistic non-membership degree, respectively. By using the above mentioned methods, we convert data of main pollution factors of air quality into normalized values in the first step. In the next step we construct the term-sets, taken into account the threshold values. Then, carrying out some calculations according to the proposed methods, in the end we get crisp values. The obtained and analyzed results from intuitionistic linguistic aggregation of air quality index illustrate the state of air quality in Azerbaijan for some years.

Keywords: air quality; intuitionistic fuzzy sets; intuitionistic linguistic number; intuitionistic linguistic weighted average.
UDC Number(s): UDC 504.05/.06; UDC 502.3

1. INTRODUCTION

In this paper with the purpose to analyze the air quality in the country, an approach with the application of intuitionistic fuzzy linguistic tools is proposed. As known air quality is one of the main factors of environmental sustainability. Environmental sustainability Indicators are measured taking into account its different aspects. Intuitionistic Fuzzy Set (IFS) is a very powerful tool for processing uncertain information. IFS are characterized by the degree of membership and the degree of non-membership. Similar to IFS, linguistic intuitionistic fuzzy set (LIFS) is characterized by a linguistic membership degree and a linguistic non-membership degree, respectively. By using the above mentioned methods, we convert data of main pollution factors of air quality into normalized values in the first step. In the next step we construct the term-sets, taken into account the threshold values. Then, carrying out some calculations according to the proposed methods, in the end we get crisp values. The obtained and analyzed results from intuitionistic linguistic aggregation of air quality index illustrate the state of air quality in Azerbaijan for some years.

Ecology problems are one of the important and influential factors in sustainable development of the states. According to the results of the calculation of the Yale and Columbia Universities of the United States, Azerbaijan in 2016 according to the Environmental Condition Index took a good 31st position among 180 countries of the world. But in 2018 it lost its position a little, and in 2020 it dropped to 72nd position [1]. According to the calculations made in the article, Azerbaijan in 2016 also ranked 1st among countries such as Russia, Turkey, Georgia and Iran [2]. There are some sub-factors of ecology. One of the main sub-factor is an air quality.Urban air pollution has a significant impact on the health of people in Europe, Asia and other, especially in the eastern parts of the European Region and in the countries with high development rate (WHO - World Health Organization). At the same time, air quality monitoring, control and management systems are not always effective enough, their legal, organizational and technical aspects require modernization.

PROBLEM STATEMENT

There are Statistic data of the main air pollution’s factors of mobile sources of Azerbaijan below (tab.1):

Table 1 Data of main pollution sub-factors of Azerbaijan [3]

	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
sulphurdioxide SO_2	528,3	554,7	604,7	655,6	700,3	**708,1**	678,9	695,6	687,3	666,2	**445,7**
nitrogenoxides N_2O	**62,3**	65,4	71,3	76,6	91,1	92,2	105,6	92,6	87	**108,8**	75,2
hydrocarbons CH_4	**101,7**	106,7	116,3	127,5	151,8	154,5	**166,7**	161,3	156,4	149	130,2
carbonmonoxide CO_2	**101,1**	106,0	115,5	126,7	150,9	153,5	**165,8**	160,9	155,4	148,7	129,9

Further we propose a method, using intuitionistic linguistic fuzzy number, by which we assess the air quality and analyze the situation.

An intuitionistic linguistic number (***ILN)*** A in X is defined [4] as (1):

$$A = \{\langle x[h_{\theta(x)}, \langle \mu_A(x), \nu_A(x)\rangle]\rangle | x \in X\} \tag{1}$$

here $h_{\vartheta(x)} \in \bar{S}$, and $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and non-membership degree of the element x related to linguistic index $h_{\theta(x)}$, respectively.

$0 \le \mu_A(x) + \nu_A(x) \le 1$, for all $x \in X$. For each $\boldsymbol{ILN}A$ in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \forall x \in X \tag{2}$$

then $\pi_A(x)$ is called the indeterminacy degree or hesitation degree of x to linguistic index $h_{\vartheta(x)}$.

In order to carry out some calculations in the proposed method, we go to the next steps for assessment air quality of the country. In the first step we normalized the statistical data. Normalised indicators are converted into intuitionistic fuzzy numbers using the intuitionistic fuzzy triangular functions *iftrif* [5] by next formulas (3):

$$Norm(x_i) = \frac{x_i - x_{min}}{x_{max} - x_{min}} \tag{3}$$

Table 2. Normalized values

	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
SO_2	0,315	0,415	0,606	0,799	0,970	1	0,889	0,952	0,921	0,840	0
N_2O	0	0,067	0,194	0,308	0,619	0,643	0,931	0,652	0,531	1	0,277
CH_4	0	0,077	0,225	0,397	0,771	0,812	1	0,917	0,842	0,728	0,439
CO_2	0	0,076	0,224	0,399	0,769	0,810	1	0,924	0,839	0,736	0,445

In the next second step we construct the term-sets (tab.3)

Table 3. Term-set

Low (L)			Medium (M)			High (H)		
0	0,18	0,36	0.33	0,51	0,69	0,66	0,83	1

Then calculates Intuitionistic fuzzy triangular membership and non-membership function of A in corresponded term-set by the formulas (4), (5) [6]:

$$\mu_{\tilde{x}}(x) = \begin{cases} \frac{u_{\tilde{x}}(x-\underline{t})}{t-\underline{t}} & if \underline{t} \leq x < t \\ u_{\tilde{x}} & if\ x = t \\ \frac{u_{\tilde{x}}(\overline{t}-x)}{\overline{t}-t} & if t < x \leq \overline{t} \\ 0 & if x < \underline{t}\ or\ x > \overline{t} \end{cases} \tag{4}$$

$$\nu_{\tilde{x}}(x) = \begin{cases} \frac{[t-x+w_{\tilde{x}}(x-\underline{t})]}{t-\underline{t}} & if \underline{t} \leq x < t \\ w_{\tilde{x}} & if\ x = t \\ \frac{[x-t+w_{\tilde{x}}(\overline{t}-x)]}{\overline{t}-t} & if\ t < x \leq \overline{t} \\ 1 & if\ x < \underline{t}\ or\ x > \overline{t} \end{cases} \tag{5}$$

where $\underline{t}, t, \overline{t}$ are the vertices of a triangular fuzzy number.

For calculating membership and non-membership function are used reduction coefficients$(u_{\tilde{x}}, w_{\tilde{x}})$, which take into account accuracy of statistical information. Using formulas (4) and (5) we calculated IFS for data (tab.2), and introduce in (tab.4).

Table 4. Values of membership and non-membership functions

		2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
SO_2	μ	0,107	0,403	0,397	0,699	0,149	0	0,556	0,238	0,396	0,798	0
	ν	0,761	0,498	0,556	0,218	0,834	1	0,378	0,734	0,557	0,109	1
	π	0,132	0,099	0,047	0,082	0,018	0	0,066	0,028	0,047	0,094	0
N_2O	μ	0	0,315	0,786	0,248	0,334	0,222	0,344	0,181	0,750	0	0,390
	ν	1	0,648	0,122	0,723	0,627	0,752	0,615	0,797	0,162	1	0,564
	π	0	0,037	0,093	0,029	0,039	0,026	0,041	0,021	0,088	0	0,046
CH_4	μ	0	0,363	0,639	0,316	0,554	0,762	0	0,415	0,792	0,339	0,512
	ν	1	0,594	0,286	0,647	0,381	0,149	1	0,536	0,115	0,622	0,428
	π	0	0,043	0,075	0,037	0,065	0,090	0	0,049	0,093	0,040	0,060
CO_2	μ	0	0,356	0,644	0,310	0,547	0,749	0	0,380	0,804	0,381	0,542
	ν	1	0,599	0,280	0,653	0,389	0,163	0,170	0,575	0,102	0,575	0,394
	π	0	0,042	0,076	0,037	0,064	0,088	0,830	0,045	0,095	0,049	0,064

In the next step we define θ (Tetta) – values of every factor in every year.

Table 5. Tetta in crisp values

	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
SO_2	1	1	2	3	3	3	3	3	3	3	1
N_2O	1	1	1	1	2	2	3	2	2	3	1
CH4	1	1	1	2	3	3	3	3	3	3	2
CO_2	1	1	1	2	3	3	3	3	3	3	2

Table 6. Tetta in linguistic values

	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
SO_2	L	L	M	H	H	H	H	H	H	H	L
N_2O	L	L	L	L	M	M	H	M	M	H	L
CH_4	L	L	L	M	H	H	H	H	H	H	M
CO_2	L	L	L	M	H	H	H	H	H	H	M

On the next stage we calculated the weight’s values of every factor in every year. Weights of indicators are estimated as follow as proposed by Boran et.al formula (6) [7]

$$\lambda_k = \frac{(\mu_k + \pi_k\left(\frac{\mu_k}{\nu_k}\right))}{\sum_{k=1}^{l}(\mu_k + \pi_k\left(\frac{\mu_k}{\nu_k}\right)} \tag{6}$$

and $\sum_{k=1}^{l} \lambda_k = 1$

Table 7. Weight's values of pollution factors

	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
SO_2	1	0,304	0,125	0,512	0,085	0	0,640	0,191	0,094	0,660	0
N_2O	0	0,209	0,272	0,137	0,134	0,088	0,360	0,144	0,099	0	0,171
CH_4	0	0,245	0,156	0,178	0,245	0,469	0	0,350	0,123	0,160	0,237
CO_2	0	0,242	0,238	0,174	0,356	0,443	0	0,316	0,339	0,181	0,385

In the next step we calculated intuitionistic linguistic weighted average (ILWA) estimated meaning of sub-indices by the following formula and analyzed situation:

$$\boldsymbol{ILWA} = \langle S_{\sum_{k=1}^{t} \lambda_k \theta\left(a_{ij}^{k}\right)}, (1 - \prod_{k=1}^{t} (1 - \mu\left(a_{ij}^{k}\right))^{\lambda_k}, \prod_{k=1}^{t} (\nu\left(a_{ij}^{k}\right))^{\lambda_k} \rangle \quad (7)$$

$$ILWA\ (Air\ 2010) = S_1\langle 0.107, 0.761\rangle - L;$$

$ILWA(Air\ 2011) = S_1\langle 0.365, 0.575\rangle - L;$
$ILWA(Air\ 2012) = S_{1.13}\langle 0.689, 0.219\rangle - L - M;$
$ILWA(Air\ 2013) = S_{2.38}\langle 0.545, 0.377\rangle - M - H;$
$ILWA\ (Air\ 2014) = S_{2.31}\langle 0.487, 0.453\rangle - M - H;$
$ILWA(Air\ 2015) = S_{2.91}\langle 0.729, 0.179\rangle - M - H;$
$ILWA(Air\ 2016) = S_{3.00}\langle 0.489, 0.451\rangle - H;$
$ILWA(Air\ 2017) = S_{2.86}\langle 0.343, 0.616\rangle - M - H;$
$ILWA(Air\ 2018) = S_{2.56}\langle 0.764, 0.139\rangle - M - H;$
$ILWA(Air\ 2019) = S_{3.00}\langle 0.702, 0.193\rangle - H;$
$ILWA(Air\ 2020) = S_{1.54}\langle 0.496, 0.445\rangle L - M.$

4. DISCUSSION AND CONCLUSIONS

The results obtains show unsatisfactory Air Quality during some years in Azerbaijan. But in last year the Air Quality increased some. After the Conference by Global Climate Changes in the World the Government of Azerbaijan has pledged to strengthen measures by improve the Ecological state and the Air Quality in the next ten-years

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A PROBLEM RELATED WITH JACKSON-BERNSTEIN CLASSIC THEOREM FOR PERIODIC FUNCTIONS

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This paper is devoted to the analogue of Jackson-Bernstein classic theorem on the curves in the complex plane in integral metric. In a uniform metric this problem is close to terminal solution. Similar problem on the curves in a complex plane in integral metric was studied not enough and, in such formulation, it is considered for the first time.

Keywords: Jackson-Bernstein theorem; closed curve; polynomial approximation; continuity modulus; complex plane.

Introduction

In the present paper, direct theorems of polynomial approximation on closed curves in a complex plane in the metric L_p are considered.

We’ll need some results from [2] where it is given survey of basic of basic stages of investigations related with J. Walsh problem in the metric C.

Let the function $u(z)$ be given on a closed rectifiable curve Γ and $L_p(\Gamma,u)$ denote a weight space of functions f with weight u and norm

$$\|f\|_{L_p(\Gamma,u)} = \left(\int\limits_{\Gamma} |f(z)u(z)|^p \, |dz| \right)^{1/p} < \infty \quad (p \geq 1).$$

Let $K(0,a]$ be a set of positive non-decreasing functions determined on $(0,a]$ (a is an arbitrary positive number) and $P(0,a]$ be a set of positive non-increasing functions on $(0,a]$. By $K_0(0,a]$ we denote a set of those functions $f \in K(0,a]$ for which $f(t) \to 0$ as $t \to 0$.

Let the function $\varphi(\widetilde{\varphi})$ map the exterior (interior) of a curve Γ onto the exterior (interior) of a unit circle γ_0 and $\psi\ (\widetilde{\psi})$ be a function inverse to $\varphi\ (\widetilde{\varphi})$.

Assuming Γ to be smooth or piecewise-smooth and $f \in L_p(\Gamma)$ we consider the quantities (generalized modulus of continuity):

$$z_h = \psi\left(\varphi(z)e^{ih}\right), \ \widetilde{z}_h = \widetilde{\psi}\left(\widetilde{\varphi}(z)e^{ih}\right);$$

$$u_p(\delta) = u_p(f,\delta)_\Gamma = \sup_{|h| \leq \delta} \| f(z_h) - f(z) \|_{L_p(\Gamma)}$$

and

$$\tilde{u}_p(\delta)=\tilde{u}_p(f,\delta)_\Gamma=\sup_{|h|\le\delta}\|f(\tilde{z}_h)-f(z)\|_{L_p(\Gamma)}.$$

Obviously, these quantities possess all the properties of ordinary modulus of continuity:

1^0. $\tilde{u}_p(\delta),\ u_p(\delta)\in K_0(0,l]$, where l is the length of the curve Γ;

2^0. The functions u_p and $\tilde{u}_p$ are semiadditive i.e.

$u_p(\delta_1+\delta_2)\le u_p(\delta_1)+u_p(\delta_2)$ and $\tilde{u}_p(\delta_1+\delta_2)\le\tilde{u}_p(\delta_1)+\tilde{u}_p(\delta_2)$ for $\delta_1,\delta_2\in(0,l]$;

3^0. $u_p,\ \tilde{u}_p\in C[0,l]$, i.e., they are continuous on $[0,l]$.

It is easy to show that, on the curves Γ, for which $|\psi'(\omega)|\asymp 1$ or $\psi(\omega)\in H'(\gamma)$, ($H'(\gamma)$ is a Holder class) the modulus of continuity of the form $u_p(\delta)=u_p(f,\delta)_\Gamma$ are equivalent as $\delta\to 0$ to modulus of continuity introduced by S.Ya. Alper:

$$\omega^*{}_p(f,\delta)_\Gamma=\omega_p(f_0,\delta)=\sup_{|h|\le\delta}\|f_0(\omega e^{ih})-f_0(\omega)\|_{L_p(\gamma_0)},$$

where $f_0(\omega)=f(\psi(\omega))$, and also to the continuity modulus

$$\omega_p(\delta)=\omega_p(f,\delta)=\sup_{|h|\le\delta}\|f(z(s+h))-f(z(s))\|_{L_p(\Gamma)},$$

where $z=z(s)$ is a parametric equation of the curve Γ.

In the case when the curve Γ belongs to a more wide class of curves, the fulfillment of condition 2^0 for the function $u_p(\tilde{u}_p)$ is not always possible, when condition 1^0 is fulfilled. In this connection, let’s introduce the following ritualized integral continuity modulus whose construction belongs to S.B. Stechkin:

$$\omega_p(\delta)=\omega_p(f,\delta)_\Gamma=\delta\sup_{t\ge\delta}t^{-1}u_p(f,t)_\Gamma \text{ and}$$

$$\tilde{\omega}_p(\delta)=\tilde{\omega}_p(f,\delta)_\Gamma=\delta\sup_{t\ge\delta}t^{-1}\tilde{u}_p(f,t)_\Gamma.$$

Obviously for these quantities the following statements hold:

a) $u_p(\delta)\le\omega_p(\delta)$ and $\tilde{u}_p(\delta)\le\tilde{\omega}_p(\delta)$;

b) $\omega_p(\delta)\in K_0(0,l]$ and $\tilde{\omega}_p(\delta)\in K_0(0,l]$;

c) $\delta^{-1}\omega_p(\delta)\in P(0,l]$ and $\delta^{-1}\tilde{\omega}_p(\delta)\in P(0,l]$;

d) if $\delta^{-1}g(\delta)\in P(0,l]$ and $u_p(\delta)\le g(\delta)$ $(\tilde{u}_p(\delta)\le g(\delta))$, then $\omega_p(\delta)\le g(\delta)$ $(\tilde{\omega}_p(\delta)\le g(\delta))$;

e) if $u_p(\widetilde{u}_p)$ is semiadditive, then $u_p(\delta) \asymp \omega_p(\delta) \quad (\widetilde{u}_p(\delta) \asymp \widetilde{\omega}_p(\delta))$*.

Notice, that the statement d) shows that the function $\omega_p(\widetilde{\omega}_p)$ is the best majorant for $u_p(\widetilde{u}_p)$ on all the class of functions $g(\delta)$, for which $\delta^{-1}g(\delta) \in P(0,l]$.

Basic definitions and notations

Let's introduce the following regularized generalized integral smoothness modulus:

$$\omega_p^{(2)}(f,\delta)_\Gamma = \delta^2 \sup_{t \geq \delta} t^{-2} u_p^{(2)}(f,t)_\Gamma,$$

where $u_p^{(2)}(f,t)_\Gamma = \sup_{|h| \leq t} \| f(z_h) + f(z_{-h}) - 2f(z) \|_{L_p(\Gamma)}$,

and also

$$\widetilde{\omega}_p^{(2)}(f,\delta)_\Gamma = \delta^2 \sup_{t \geq \delta} t^{-2} \widetilde{u}_p^{(2)}(f,t)_\Gamma,$$

where $\widetilde{u}_p^{(2)}(f,t)_\Gamma = \sup_{|h| \leq t} \| f(\widetilde{z}_h) + f(\widetilde{z}_{-h}) - 2f(z) \|_{L_p(\Gamma)}$.

It is easy to see that $\omega_p^{(2)}(f,\delta)$ and $\widetilde{\omega}_p^{(2)}(f,\delta)$ possess all the properties of ordinary smoothness modulus and they are the best majorants for the functions $u_p^{(2)}(f,\delta)_\Gamma$ and $\widetilde{u}_p^{(2)}(f,\delta)_\Gamma$ respectively, among all the functions of smoothness modulus type.

Notice that for the cited generalized smoothness modulus the relations:

$$\omega_p^{(2)}(f,\eta) \leq \delta^{-2}(\delta+\eta)^2 \omega_p^{(2)}(f,\delta) \quad \forall \delta > 0, \forall \eta > 0,$$

$$\widetilde{\omega}_p^{(2)}(f,\eta) \leq \delta^{-2}(\delta+\eta)^2 \widetilde{\omega}_p^{(2)}(f,\delta) \quad \forall \delta > 0, \forall \eta > 0$$

and

$$\eta^{-2}\omega_p^{(2)}(f,\eta) \leq 4\delta^{-2}\omega_p^{(2)}(f,\delta) \qquad \text{for} \qquad 0 < \delta < \eta \ ,$$

$$\eta^{-2}\widetilde{\omega}_p^{(2)}(f,\eta) \leq 4\delta^{-2}\widetilde{\omega}_p^{(2)}(f,\delta) \qquad \text{for} \qquad 0 < \delta < \eta .$$

hold as well.

By $M(G)$ we denote a class of functions, analytical and bounded everywhere in the domain G and bounded almost everywhere (in linear sense) on ∂G.

Let $f \in M(G)$ and $F_h(z) = f(z_h) + f(z_{-h}) - 2f(z)$, where $h \in [0,\pi]$, $z_h = \psi(\varphi(z)e^{ih})$ and

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* The relations $A \asymp B$ and $A \preccurlyeq B$ $(A > 0, B > 0)$ each time are determined with respect to some fixed collection of parameters and correspond to the inequalities $C_1 B \leq A \leq C_2 B$ and $A \leq C_3 B$, where $C_k > 0$ $(k = 1,2,3)$ are the constants independent of the mentioned collection of parameters.

$$J\,F_h=\left(J\,F_h\right)_\Gamma(t)=\frac{1}{\pi i}\int_\Gamma\frac{F_h(z)}{z-t}dz,\qquad t\in\Gamma. \tag{1}$$

We’ll say that a Jordan rectifiable curve Γ belongs to the class M (or $\Gamma\in M$) if for any $f\in M(\overline{G})$ integral (1) exists almost everywhere on Γ and for each $p\geq 1$ it holds the estimate

$$\|J\,F_h\|_{L_p(\Gamma)}\leq C(p)\|F_h\|_{L_p(\Gamma)},\quad \forall h\in[0,\pi],$$

where $C(p)$ is a constant dependent only on the indicated parameter p.

Notice that such a definition of a class of curves M allows to extend a class of curves considered earlier and also to consider the case $p=1$.

We’ll need the following auxiliary proposals.

Lemma A ([2], Lemma B). Let Γ be a closed Jordan rectifiable curve and f be a function determined almost everywhere on Γ and measurable. If the singular integral

$$Sf=(Sf)_\Gamma(t)=\frac{1}{\pi i}\oint_\Gamma\frac{f(z)}{z-t}dz,\quad t\in\Gamma,$$

exists almost everywhere on Γ, then the Cauchy type integral

$$\Phi(\xi)=\frac{1}{\pi i}\int_\Gamma\frac{f(z)}{z-\xi}dz,\quad \xi\overline{\in}\Gamma,$$

almost everywhere on Γ has definite corner values equal

$$\Phi^{\pm}(t)=\pm\frac{1}{2}f(t)+\frac{1}{2}Sf(t). \tag{2}$$

Conversely, if the Cauchy type integral Φ has almost everywhere on Γ corner boundary values from within and from outside of Γ, then almost everywhere on Γ there exists a particular (singular) integral Sf and boundary values of Cauchy type integral Φ is expressed by the formula (2).

Lemma B ([2], Lemma Г**).** For representability of the function f by the Cauchy integral by its boundary values, it is necessary and sufficient that $f\in E_1(G)$ ($E_p(G)$ is V.I. Smirnov's class).

Lemma C. ([2], Lemma 2). Let $G\in S$, $f\in E_p(G)$ $(p>0)$ and $\{f(t);\,t\in\Gamma\}$ be boundary values of the function $f(z)$, $z\in G$. Then there will be found a sequence of functions $\{\Phi_k(t)\}$ being boundary values of bounded analytic in G functions $\Phi_k(z)$, such that almost for all $t\in\Gamma$ $\lim\limits_{k\to\infty}\Phi_k(t)=f(t)$ and $|\Phi_k(t)|\leq|f(t)|$ $(k=1,2,\ldots)$.

Theorem 1. ([2]). Under the hypotheses of lemma C there exists a sequence of functions $\{\Phi_k{}^*\}$, such, that:

a) $\lim\limits_{k\to\infty}\int\limits_{\Gamma} \left|f(t)-\Phi_k{}^*(t)\right|^p |dz| = 0$;

b) The functions $\Phi_k{}^*$ are bounded on Γ and coincide almost everywhere on Γ with boundary values of analytic functions bounded in G;

c) $\lim\limits_{k\to\infty}\int\limits_{\Gamma} \left|f(t_{\pm h})-\Phi_k{}^*(t_{\pm h})\right|^p |dt| = 0$ where $t_{\pm h}\in\Gamma$ $\left(t_{\pm h}=\psi\left(\varphi(t)e^{\pm ih}\right)\right)$,

and the functions ψ and φ conformally map the exterior (or interior) of a unit circle onto $G^+\left(G^-\right)$ and vice versa;

d) $\lim\limits_{k\to\infty} u_p^{*(2)}\left(\Phi_k^*,h\right)\asymp u_p^{*(2)}(f,h)_\Gamma$, $\left(\lim\limits_{k\to\infty}\tilde{u}_p^{*(2)}\left(\Phi_k^*,h\right)\asymp u_p^{*(2)}(f,h)_\Gamma\right)$,

where $u_p^{*(2)}(g,h)=\left\|g(t_h)+g(t_{-h})-2g(t)\right\|_{L_p(\Gamma)}$.

By $E_p^*(G)$ $(p\ge 1)$ we denote a class of functions $f\in E_p(G)$ with a finite regularized smoothness modulus $\omega_p^{(2)}(\delta)=\omega_p^{(2)}(f,\delta)_\Gamma$.

Main results

Theorem 2. If the curve $\Gamma\in M$ and $f\in E_p^*(G)$ $(p\ge 1)$, then for each natural n there exists a polynomial P_n such that

$$\left\|f-P_n\right\|_{L_p(\Gamma)}\le const\,\omega_p^{(2)}\left(f,\frac{1}{n}\right)_\Gamma .$$

Proof. Let's consider a sequence of the functions $\{f_k(z)\}$ from the class $M\left(\overline{G}\right)$ for which the following relations are fulfilled (see Theorem 1)

$$\lim_{k\to\infty}\int_{\Gamma} \left|f(t)-f_k(t)\right|^p |dt| = 0 , \tag{3}$$

$$\lim_{k\to\infty}\int_{\Gamma} \left|f(t_{\pm h})-f_k(t_{\pm h})\right|^p |dt| = 0 , \tag{4}$$

$$\lim_{k\to\infty}\left\|f_k(t_h)+f_k(t_{-h})-2f_k(t)\right\|_{L_p(\Gamma)}=\left\|f(t_h)-f(t_{-h})-2f(t)\right\|_{L_p(\Gamma)}. \tag{5}$$

Now, considering $\Gamma\in M$ and considering that by the hypotheses of the theorem $f(z_{\pm h})\in L_p(\Gamma)$, we can affirm, that the integral

$$B_k=\int_{\Gamma}\frac{f_k(z_h)-f_k(z_{-h})}{z-t}dt,\quad \forall t\in\Gamma ,$$

exists on the sense of the chief value almost everywhere on Γ. This allows us to approximate the functions $f_k(z)$ by polynomials (see [3]), that for $t\in\Gamma$

are representable in the following form

$$P_{n,k}(t)=\frac{1}{4\pi^2 i}\int_0^{\pi} K_n(h)dh\int_{\Gamma} \frac{f_k(z_h)+f_k(z_{-h})}{z-t}dz+\frac{1}{4\pi i}\int_0^{\pi} K_n(h)[f_k(t_h)+f_k(t_{-h})]dh, \quad (6)$$

where $K_n(h)$ is a positive kernel representing itself as a trigonometric polynomial of power not higher than n and satisfying the conditions:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(h)dh=1 \qquad (n=0,1,2,\ldots), \quad (7)$$

$$\int_{-\pi}^{\pi} |K_n(h)|dh\le C_1 \qquad (n=0,1,2,\ldots), \quad (8)$$

$$\int_{-\pi}^{\pi}|t|^k |K_n(h)|dt\le C_2(k)(n+1)^{-k} \qquad (n=0,1,2,\ldots), \quad (9)$$

where C_1 and C_2 are positive constants. Notice that to these conditions, in particular, satisfy the Jackson kernels (see [3]). Besides, from the fulfillment of conditions (7)-(9) we immediately get (see [3]):

$$\int_{-\pi}^{\pi}\left(|t|+\frac{1}{n}\right)^k |K_n(h)|\, dt\le C_3(k)\,(n)^{-k} \qquad (n=1,2,\ldots) \quad (10)$$

Since the function $f_k\in M(\overline{G})\subset E_1(G)$, by lemma B, we can represent them by the Cauchy integral:

$$f_k(z)=\frac{1}{2\pi i}\int_{\Gamma} \frac{f_k(\xi)}{\xi-z}d\xi, \quad \forall z\in G.$$

This allows us, by lemma A for the function $f_k(t)$ to get the following representation

$$f_k(t)=\frac{1}{2}f_k(t)+\frac{1}{2\pi i}\int_{\Gamma}\frac{f_k(\xi)}{\xi-t}d\xi, \quad t\in\Gamma,$$

almost everywhere on Γ.

By relation (7), we can represent the last expression in the form

$$f_k(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(h)\,f_k(t)dh+\frac{1}{2\pi^2 i}\int_{-\pi}^{\pi} K_n(h)dh\int_{\Gamma}\frac{f_k(\xi)}{\xi-t}d\xi. \quad (11)$$

Now, having used relations (6) and (11) we estimate norm of the difference $f_k(t)-P_{n,k}(t)$ in the metric $L_p(\Gamma)\ (p>1)$. We have

$$\|f_k-P_{n,k}(t)\|_{L_p(\Gamma)}=\left\|\frac{1}{4\pi^2 i}\int_0^{\pi} K_n(h)dh\int_{\Gamma}\frac{f_k(z_h)+f_k(z_{-h})-2f_k(z)}{z-t}dz+\right.$$

$$+\frac{1}{4\pi}\int_0^{\pi} K_n(h)\left[f_k(t_h)+f_k(t_{-h})-2f_k(t)\right]dh\Bigg\|_{L_p(\Gamma)} \leqslant$$

$$\leqslant\left\|\int_0^{\pi} K_n(h)dh\int_{\Gamma}\frac{f_k(z_h)+f_k(z_{-h})-2f_k(z)}{z-t}dz\right\|_{L_p(\Gamma)}+$$

$$+\left\|\int_0^{\pi} K_n(h)\left[f_k(t_h)+f_k(t_{-h})-2f_k(t)\right]dh\right\|_{L_p(\Gamma)}.$$

Further, applying generalized Minkovsky inequality we get:

$$\left\|f_k-P_{n,k}(t)\right\|_{L_p(\Gamma)}\leqslant\int_0^{\pi}|K_n(h)|dh\left\|\int_{\Gamma}\frac{f_k(z_h)+f_k(z_{-h})-2f_k(z)}{z-t}dz\right\|_{L_p(\Gamma)}+$$

$$+\int_0^{\pi}|K_n(h)|\left\|f_k(t_h)+f_k(t_{-h})-2f_k(t)\right\|_{L_p(\Gamma)}\leqslant$$

$$\int_0^{\pi}|K_n(h)|A_{k,p}(h)dh+\int_0^{\pi}|K_n(h)|U_{k,p}^{*(2)}(h)dh \tag{12}$$

where

$$A_{k,p}(h)=\left(\int_{\Gamma}\left|\int_{\Gamma}\frac{f_k(z_h)+f_k(z_{-h})-2f_k(z)}{z-t}dz\right|^p|dt|\right)^{1/p},$$

$$U_{k,p}^{*(2)}=\left\|f_k(t_h)+f_k(t_{-h})-2f_k(t)\right\|_{L_p(\Gamma)}.$$

In view of the fact that $\Gamma\in M$, obviously, we have

$$A_{k,p}(h)\prec U_{k,p}(h) \tag{13}$$

Hence, we get:

$$\left\|f_k-P_{n,k}\right\|_{L_p(\Gamma)}\leqslant\int_0^{\pi}|K_n(h)|U_{k,p}(h)dh \tag{14}$$

Besides, it is easy to notice that the sequence $\{P_{n,k}\}$ for fixed n tends to some polynomial P_n as $k\to\infty$, i.e.

$$\lim_{k\to\infty}\left\|P_{n,k}-P_n\right\|_{L_p(\Gamma)}=0 \tag{15}$$

Now, let’s return to the function $f\in E_p^*(G)$ for which the sequence $\{f_k\}$ was constructed and that satisfies conditions (3) and (5). Obviously, we have:

$$\|f - P_n\|_{L_p(\Gamma)} \le \|f - f_k\|_{L_p(\Gamma)} + \|f_k - P_{n,k}\|_{L_p(\Gamma)} + \|P_n - P_{n,k}\|_{L_p(\Gamma)}.$$

By (14) the last estimate is written in the form

$$\|f - P_n\|_{L_p(\Gamma)} \le \|f - f_k\|_{L_p(\Gamma)} + \|P_n - P_{n,k}\|_{L_p(\Gamma)} + \int_0^{\pi} |K_n(h)| U_{k,p}(h)dh \cdot$$

Now, considering (3)-(5) and using Helly theorem (see [4], p.366) on passage to the limit under the Lebesgue-Stielties integral sign, we get

$$\|f - P_n\|_{L_p(\Gamma)} \le \int_0^{\pi} |K_n(h)| U_p^*(h)dh,$$

where $U_p^*(h) = \|f(z_h) + f(z_{-h}) - 2f(z)\|_{L_p(\Gamma)}$.

Hence, it immediately follows the estimate

$$\|f - P_n\|_{L_p(\Gamma)} \le \int_0^{\pi} K_n(h)\omega_p^{(2)}(f,h)_\Gamma\, dh \cdot$$

Hence, by the known reasonings we get:

$$\|f - P_n\|_{L_p(\Gamma)} \lessdot \omega_p^{(2)}\left(f + \frac{1}{n}\right)_\Gamma .$$

In fact, we estimate the expression

$$\int_0^{\pi} |K_n(t)| \omega_p^{(2)}(f,t)_\Gamma\, dt = \int_0^{\pi} |K_n(t)| \vartheta(t)dt, \quad \text{where } \vartheta(t) \overset{df}{=} \omega_p^{(2)}(f,t)_\Gamma \cdot$$

Using the properties of smoothness modulus $\omega_p^{(2)}(f,\delta)_\Gamma$, we can show the validity of the following relation

$$\vartheta(|t|) \le n^2\left(|t| + \frac{1}{n}\right)^2 \vartheta\left(\frac{1}{n}\right).$$

Hence and from inequality (10) we get:

$$\int_0^{\pi} |K_n(t)|\, \vartheta(t)dt \le n^2\vartheta\left(\frac{1}{n}\right)\int_0^{\pi} \left(t + \frac{1}{n}\right)^2 |K_n(t)|dt \le n^2\vartheta\left(\frac{1}{n}\right)n^{-2} = \vartheta\left(\frac{1}{n}\right),$$

Q.E.D.

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PRINCIPAL COMPONENT ANALYSIS FOR ESTIMATING OF SUSTAINABILITY OF SOSIO-ECONOMIC DEVELOPMENT

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The criteria for sustainable development include the requirement to minimize waste and the condition that environmental pollution in the future would not exceed its current level. Taking into account these criteria will preserve the environment for future generations and will not worsen the ecological living conditions of society today. To analyze the sustainability of socio-economic development, the classification of indicators proposed by the UN Commission on Environmental Protection is used. According to this classification, more than 130 indicators considered are divided into 3 groups: economic, social and enviromental. Some of these indicators are difficult to study due to the lack of statistical data, but even the availability of data on most of them makes it difficult to analyze them together. Therefore, the main goal of principal components analysis is to reduce the number of variables by classifying variables and identifying relationships between them. The reduction is achieved by highlighting hidden (latent) common factors that explain the connections between the observed features of the object.

Keywords: sustainable development, socio-economic development, principal components analysis, latent factors.

1. INTRODUCTION

At present, humanity, realizing the real threat of a global ecological catastrophe, has come to a common opinion on the development of a concept for its further development. This concept has become the concept of sustainable development. Sustainable development is progress that meets the needs of the present, but does not forget about needs of the future generations. Ensuring environmental safety of socio-economic development requires a transition to a sustainable type of economic development that respects the compromise between current and future consumption. The most difficult problem of the transition to sustainable development is finding optimal ways and means of combining the tasks of economic development and ensuring the ecological security of human existence. As many experts in this field have emphasized, the main obstacle to sustainable development lies in the contradiction between the market nature of economic processes and the non-market nature of actions to protect the environment. Compliance with the basic criteria of sustainable development requires that the amount of renewable resources would not decrease over time, and as for non-renewable resources, it is assumed that the rate of their depletion will slow down. In addition, the criteria for sustainable development include the requirement to minimize waste and the condition that environmental pollution in the future would not exceed its current level. Taking into account these criteria will preserve the environment for future generations and will not worsen the ecological living conditions of society today. Though, sustainable socio-economic development should be refered taking into account economic, social and ecological factors [1]. That’s why to research the numerous indicators of socio-economic development systematically, these indicators have been groupped and analysed [2]. The necessity for assessment of sustainability of soci-economic development of the country shows the actuality of the research.

2. The necessaty of application principal components analysis for estimating of sustainability of socio-economic development

In connection with the transition of most countries of the world to a sustainable type of development, the question arose about how to assess Azerbaijan's progress in this direction and what macroeconomic indicators to use to analyze the sustainability of the country's economic development. To analyze the sustainability of socio-economic development, the classification of indicators proposed by the UN Commission on Environmental Protection is used. According to this classification, more than 130 indicators considered are divided into 3 groups: economic, social and environmental. The issues of building a system of indicators of sustainable socio-economic development were considered in the works of many foreign and domestic scientists. Among them, the study by V.N. Tamashevich [3], in which the system of indicators of socio-economic development can be divided into three groups, named by him as:

1) volumetric(quantitative) indicators of economic growth;
2) qualitative indicators of economic growth;
3) indicators of sustainability of socio-economic development.

Such a division of indicators, according to the author, is conditional, since some indicators can be assigned to different groups. Therefore, numerous indicators of socio-economic development to simplify their systematic research are combined into one summary group. These indicators allow to evaluate the current situation of the problem in line with reflecting the quantitative and qualitative characteristics of the problem. Some of these indicators are difficult to study due to the lack of statistical data, but even the availability of data on most of them makes it difficult to analyze them together. Therefore, the main goal of principal component analysis is to reduce the number of variables by classifying variables and identifying relationships between them. The reduction is achieved by highlighting hidden (latent) common factors that explain the connections between the observed features of the object. The main objective of research is to replace the numerous indicators of socio-economic development with several factors using principal component analysis. The task is to present these indicators as a linear combination of a relatively small number of hypothetical factors.

The principal component analysis refer to multivariate statistics which are first were investigated by an inglish scientist C.Pirson and developed by american researchers L.Gudman, Q.Hotelling, L.Terstown, K.Holzinger, Q.Kauzer later [4]. The reduction is achieved by highlighting hidden (latent) common factors that explain the connections between the observed features of the object [5]. At the first stage, the values of the considered stability indicators Xi j are standardized, i.e. their z_{ij} values are calculated, where i=1, ...m, m is the number of indicators, , j=1, ..., n, n is the number of observed years:

$z_{ij} = \frac{x_{ij} - \bar{x}}{\sigma_j}$. Later, R correlation matrix is calculated by using of standardized estimation: $R = \frac{1}{n} ZZ'$.

Eigenvalues and eigenvectors of the correlation matrix are calculated from the equation below: $|R - \lambda E| = 0$.

The values of z are represented as a linear combination of several factors:

$Z_{ij} = a_{i1} p_{1j} + a_{i2} p_{2j} + \ldots + a_{ir} p_{rj}$,

where a_{ij} -constant coefficients to be determined,

and p_{1j} ,..., p_{rj} - are the values of the factors.

In matrix form, this can be written as

Z=AP, where

Z- matrix of normalized primary indicators;

A- factor mapping matrix, matrix elements are called factor loadings;

P- matrix of factors for a given period of time.

Generalized factors are determined by the correlation matrix R of standardized source data R, which is equal to R= AA΄. Knowing the matrix R, it is possible to determine the matrix A. Finding the coefficients of the matrix A leads to the selection of factors that have the main part of the variance of the indicators. In practice, the SPSS program was used to identify factors, which determines factors using the principal component analysis. The main components are characteristic vectors of the covariance matrix. The set of principal components is a convenient coordinate system, and their contribution to the overall variance characterizes the statistical properties of the principal components. Of the total number of main components, only those that have the largest share in the total variance are left. As a rule, they leave factors that determine the variance of the initial indicators by 70% or more. Since the factors are uncorrelated, they can be represented as three orthogonal axes. The SPSS program allows you to get a graphical image of the matrix of the main components. By rotating the coordinate system around its center, we obtain various solutions. Factor loads change at the same time, but the sum of their squares remains unchanged. Based on the geometric representation of the problem under consideration, the search for an unambiguous solution is called the problem of rotation of factors. For orthogonal rotation of factors in the SPSS program, the verimax method is used. As a result of the rotation of the coordinate system by the verimax method, we find such a position of the coordinate system that for each column of the matrix A would increase large factor loads and reduce small ones. Although factor loads change, the sum of their squares remains unchanged.

3. Practical realization of principal components analysis

To promote the sustainable development in the country “The state program on poverty reduction and sustainable development in Azerbaijan Republic in 2008-2015” has been adopted as it happens in all of developed countries. Development of the country is a multilateral and multifaceted process and it has been researched from social, economic and environmental points of view. The indicators of sustainability of socio-economic development of Azerbaijan Republic for the period of 2000-2018 have been found and analyzed under three groups. These indicators reflect the changes of economic, social and environmental spheres over time. The economic indicators are:

X1- the volume of GDP;
X2- the volume of industrial production;
X3- the volume of agricultural production;
X4- the volume of domestic investments;
X5- the volume of foreign investments;
X6- government income;
X7-government expenditures;
X8- retail turnover;
X9- the average nominal wages;
X10- external trade turnover;
X11- income of the population;
X12- expenditures of the population;
X13- consumer price index (CPI, %);
X14-rate of manat to 1 US dollar
;
Social:
X15- number of population;
X16- natural increase;
X17- life expectancy at birth;
X18-infant mortality under 1 year;
X19- unemployment rate (%);
X20 - the average level of pensions;
X21-saving of the population;
ecological:
X22- emissions of pollutants into the air;
X23-discharge of sewage water.

Application of the main components method is carry out by SPSS software [6]. The result of the application of principal components analysis is the decrease of 23 indicators into 3 factors.

Table 1. Factor loading matrix

	Factors		
	F1	F2	F3
GDP	0,982	0,505	
Life expectancy at birth	0,377	0,912	
Natural increase	0,855	0,389	
Infant mortality	-0,768	-0,546	-0,305
Emissions of pollutants into the air			0,811
Discharge of sewage water		-0,879	
Volume of industrial production	0,927		
Volume of agricultural production	0,858	0,515	
retail turnover	0,918	0,387	
Consumer price index	0,888	0,398	
Rate of manat to 1 US dollar	-0,828	0,525	
Consumer expenditures	0,925	0,359	
Domestic investments	0,972	0,357	
Foreign investments		0,925	0,317
External trade turnover	-0,392		0,838
The unemployment rate	-0,818		

The first factor includes with high loadings the following indicators connected to sphere of production: GDP, volume of industrial production, volume of agricultural production; in financial sector: external trade turnover, domestic investments, consumption expenditures, consumer price index, rate of manat to 1 US dollar; in social sphere: natural increase, the unemployment rate, infant mortality. This can be explained as the following: the growth of the basic economic indicators increase the expenditures of the population and domestic investments, decrease the unemployment, stimulate the population growth, decrease infant mortality. On the other side this stimulate the inflation, that is explained pozitive connection this factor with consumer price index. In generally, we can called this factor as "economic growth and social welfare".

The second factor includes with the highest factors loadings the following indicators: foreign investments, life expectancy at birth, discharge of sewage water, and infant mortality rate. Foreign investments in national economy are mainly connected with oil industry and the activities implemented in Caspean See. Application of high level waste water treatment technologies by oil companies, resulted in the decrease of the "discharge of sewage water” indicator. On the other side, this factor resuluts in creation of new jobs, decrease in unemployment, increase of welfare, decrease of infant mortality rate.

In the thirhd factor, foreign trade turnover and environmental pollution take the highest factor loading. As Azerbaijan is an oil producing country, environmental pollution is mainly connected to the energy sector. Also, the key product in foreign trade turnover is oil. The relationship between given indicators is explained with this. This factor can be called “foreign trade and environmental factor”.

4. DISCUSSION AND CONCLUSIONS

Thus, one of the goal of analysis – shortening the sign space (8 times) has been solved by expression of the large amount of elemental traits with the least amount of internal characteristics of cases. These factors explain 94% of selected general despersion. The study of these factors contributes to the determination of the main trends and forecasting of strategy in the socio-economic development of society.

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ON INTERVAL MODELING OF FUZZY UNCERTAINTY

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In the process of solving the control tasks of complex objects often have to deal with the uncertainty of the environment functioning. For example, in economics and management have to make decisions in an uncertain state of the financial assets, economic environment, and so on.

The modern design of decision-making under uncertainty is closely related to the application of fuzzy set theory which developed by American scientist L. Zade. Expert evaluates of alternatives for a variety of measure for decision-making can be represented as fuzzy sets or numbers expressed using membership functions. The theory of fuzzy sets has found its application in different fields of mathematics, biology, psychology, linguistics and other application areas.

In this paper we interested in determining of fuzzy modeling in management and other relevant science (Economic, Financial and Ecological) and we would have a vector options to efficiency and environment program.

Keywords: Decision making, Fuzzy factor, Management, Optimal investment, Fuzzy guaranteed result.

Introduction

As noted above, the modern development of decision-making under uncertainty is mainly related to the application of the fuzzy sets theory. The presence in decision-making uncertainty does not allow us to accurately assess the impact of control actions on the objective function. If uncertainty which are exists as in the system itself, so in the observations, can be represented as stochastic processes, so methods of stochastic control are applicable to such problems. However there is a relatively large class of problems, the solution of which these methods are ineffective [5], [7], [8].

Modern developments in decision making under uncertainty are associated with the application of the theory of fuzzy sets, developed by the world-famous scientist L. Zadeh. The discussed modification of this approach for solving the considered, for example, environmental and economic problems is as follows: first of all, a suitable ordinary (“clear”) mathematical model is built with the expected most probable (“clear”) parameters, then the compiled “clear” mathematical model is converted into a fuzzy one by "blurring" the parameters into intervals of possible values, that is, the parameters are represented by fuzzy numbers.

Historically, the first and most common is the probabilistic approach to deal with uncertainty. But its use is not always correct, because it requires statistical homogeneity of random events and knowledge of the distribution law, so sometimes introduced non-classical subjective probability which are not has a partial sense and expresses a person point of view who decide with a deficit of information. Source of uncertainty cannot be random and sometimes can be partially or fully deterministic. At the present time developed quantitative decision-making methods (maximization of expected utility theory of minimax, game theory, etc.) helps to choose the best solution from a set of options only in terms of one particular type of uncertainty or with full certainty. The application of the theory of probability for operating with uncertain values leads to the fact that uncertainty, regardless of its nature, is identified with the accident, while blurry or fuzzy (fuzziness) is the main source of uncertainty in many decision-making processes.

Therefore, the account of uncertainty in solving problems largely changes methods of decision making: the principle of representation of input data and model parameters changes, notion of solving the problem and the optimal solution become ambiguous [11].

The system of linear algebraic relations (equations and inequalities) is the simple stand more widely in use mathematical model of most of problems of applied and computational mathematics [6]. However, in real problems the values of coefficients and right-hand sides of such may have the indefinite even probabilistic character.

The English term “fuzzy sets” suggested by L.Zadeh [1] is visually illustrated by language examples (almost, not quite and so on) and has interesting applications on sphere of artificial intellect in the processes of construction of mathematical models of real situations. In this work the problems of identification of unknown characteristics of model in case when the coefficients of linear fuzzy relations include these characteristics. They presents evident interest in connection with problems of control by complex systems, medical diagnostics and much other ones, in which determining factors often have fuzzy characters, and another time, in generally, they are determined by subjective way.

Let give some definitions and significations which are used in present work:

R is a widen numerical straight; {x | p(x)} is the set of all x for which the condition p(x) is satisfied; the one-point set {C} ($C \in R$)we’ll interpret as interval [C,C], the ends of which are coincided; the fuzzy subset $\hat{A}$ in R we identify (see, for ex. Орловский С.А., 1981) by the totality of well regulated pairs {x, A(x)},$x \in R$, where $A:R\rightarrow[0,1]$ is the function of belonging of fuzzy defined set $\hat{A}$, and its value A(x) – by the degree of belonging of element x to $\hat{A}$; the fuzzy number (FN) is a normal fuzzy set $\hat{A}$ in R characterized by that all sets of nonnegative level $\hat{A} \overset{\text{def}}{=} \{x \mid A(x) \geq \lambda\}$ (λ [0,1]) are convex and reserved intervals; $\widehat{N}$ is family of all FN in R.

Definition 1. By interval number(IN) we understand the finite reserved interval $\alpha =[\alpha^-, \alpha^+]$ ($\alpha^- \leq \alpha^+$) over R; I(R) is the totality of all IN [3].

All IN in dependence of their structure may be divided by four groups:

$I^+(R) \stackrel{\text{def}}{=} \{ \alpha \in I(R) \mid \forall x \in \alpha: x > 0\}$;

$I^-(R) \stackrel{\text{def}}{=} \{ \alpha \in I(R) \mid \forall x \in \alpha: x < 0\}$;

$I^0(R) \stackrel{\text{def}}{=} \{ \alpha \in I(R) \mid \alpha =[\alpha^-, \alpha^+] : \alpha^-\alpha^+ =0\}$;

$I^{\pm}(R) \stackrel{\text{def}}{=} \{ \alpha \in I(R) \mid \alpha =[\alpha^-, \alpha^+] : \alpha^-\alpha^+ < 0\}$;

Further, let α, β ∈ I(R) and +, −,·,÷,∨,∧- are four arithmetical operations, the operation of taking of maximum and minimum accordingly. Then the operation (*) acted as $\alpha * \beta = \{z \mid x (*)y = z;\ x \in \alpha, y \in \beta\}$ synonymous defined a new IN γ where (*)∈ {+, −,·,÷,∨,∧} . Really, z = f(x, y) = x (∗) y is a continuous function and, therefore α(∗)β ∈ I(R) . It is not difficult to verify that ∀ k ∈ R; α, β ∈ I(R)

$\alpha + \beta =[\alpha^- + \beta^-; \alpha^+ + \beta^+]$;

$\alpha - \beta =[\alpha^- - \beta^+; \alpha^+ - \beta^-]$;

$\alpha \cdot \beta = [\alpha^-\beta^- \wedge \alpha^-\beta^+ \wedge \alpha^+\beta^- \wedge \alpha^+\beta^+;[\alpha^-\beta^- \vee \alpha^-\beta^+ \vee \alpha^+\beta^- \vee \alpha^+\beta^+]$;

$\alpha \div \beta = [\frac{\alpha-}{\beta-}\wedge\frac{\alpha+}{\beta-}\wedge\frac{\alpha-}{\beta+}\wedge\frac{\alpha+}{\beta+};\ \frac{\alpha-}{\beta-}\vee\frac{\alpha+}{\beta-}\vee\frac{\alpha-}{\beta+}\vee\frac{\alpha+}{\beta+}]$;

$\alpha\vee\beta=[\alpha^- \vee\beta^-; \alpha^+ \vee\beta^+]$; $\alpha\wedge\beta=[\alpha^- \wedge\beta^-; \alpha^+ \wedge\beta^+]$;

$k\alpha \stackrel{\text{def}}{=} [k, k][\alpha^-,\alpha^+] =[k\alpha^-,k\alpha^+]$, if $k \geq 0$; $[k\alpha^+,k\alpha^-]$ if $k< 0$; $\alpha \cong \beta \Leftrightarrow \alpha^- = \beta^-$ and $\alpha^+ = \beta^+$ $\alpha \leq \beta \Leftrightarrow \alpha^- \leq \beta^-$ and $\alpha^+ \leq \beta^+$.

Now all linear relations in IN which have form: αx = β or αx < β (α, β € I(R) are known, x€ I(R) – will be found) by method of their solving may be divided by 4 groups (in all there are $C_4^1\ C_4^1 = 16$ relations in each case).

We consider the equation

$$\alpha\, x \cong \beta\ (\alpha, \beta \in I^{\pm}(R)) \qquad (1)$$

or inequality

$$\alpha\, x \leq \beta\ (\alpha, \beta \in I^{\pm}(R)) \qquad (2)$$

Let ∀ γ € $I^{\pm}(R)$ MaxAbs γ(MinAbs γ) - that end of the interval γ which on absolute value not less (not more) other and ∀α, β€ $I^{\pm}(R)$ we define ordinary numbers m, n and d:

$$m=\frac{\text{MaxAbs } \beta}{\text{MaxAbs } \alpha};\ n=\frac{\text{MinAbs}\beta}{\text{MinAbs}\alpha};\ d = |m| - |n| .$$

On the base of above considered properties of the interval arithmetic the truth of next statements is proved.

Theorem1. The equation (1) has solution then and only then when d ≤ 0. Otherwise, this equation has not a solution. If d= 0, then (1) has infinitely number of solutions and always there is a maximal (with respect to including) element - solution. The uniqueness of the solution of equation (1) is conditioned by d <0.

Short explanation of Theorem 1: If the equation (1) has a solution (d ≤ 0), the number m, depending on its sign will coincide with one of the ends of the segment solutions m= $[x^-,x^+]$ (i.e. if m < 0 ⇒ m = x^- and, if m > 0 ⇒ m = x^-). If the solution is unique (d < 0), then the solution of (1) is [*m, l*] (when m < 0) or [l, m] (if m > 0), where l = $\frac{\text{MinAbs } \beta}{\text{MaxAbs } \alpha}$. In the case when the equation (1) has an infinite number of solutions, then X = [m, x] (when m < 0 and m ≤ x ≤ l) or X = [x, m] (with m > 0 and l ≤ x ≤ m). Finally, the case d > 0 will corresponds to the absence of solutions of equation (1).

Consider a few examples:

!) α = [− 1, 4], β = [− 2, 10]. Because d = $\frac{10}{4}$ - $\frac{2}{1}$= 0.5 > 0, then αx = β has no solution.

!!) α = [− 5, 2], β = [− 15, 6]. Then d = $\frac{15}{5}$ - $\frac{6}{2}$= 0, then αx = β has an infinite number of solutions [x^-, 3], where l = −1,2 ≤ x^- ≤ 3.

!!!) α = [− 5, 2], β = [− 15, 10]. Because d = −2 < 0, then the original equation αx = β has a unique solution X = [− 2,3] (in this case l = 3).

Theorem 2. For solving of inequality (2) it is necessary and sufficiently those relations β^-= MaxAbs β and |m| − |n| > 0 are not satisfied simultaneously.

Indeed, it is not difficult to check that if β^-= MaxAbs β and |m| − |n| ≤ 0 inequality (2) always has a solution:

- if m > 0 ⇒ x^- ∈] − ∞, l], x^+∈ [m, n];
- if m < 0 ⇒ x^- ∈ [n, m], x^+∈ [x^-, l].

In the case if |m| − |n| > 0, the relation (2) has no solutions. Next, in the case if MaxAbs = β^+, then (2) can always be solved:

- when m > 0 ⇒ x^- ∈ [k, l], x^+ ∈ [x^-, m];
- if m < 0 ⇒ x^- ∈ [m, x^-], x^+ ∈ [l, k], where k =$\frac{\text{MaxAbs } \beta}{\text{MinAbs } \alpha}$.

Further, these results are applied to approximately (with a sufficiently accuracy) solution of linear fuzzy equations

$$\alpha x + \beta = \gamma \qquad (3)$$

or inequality

$$\alpha x + \beta \le \gamma \qquad (4)$$

of more general form, where α, β, γ ∈ $\hat{N}$ are known and x ∈ $\hat{N}$ will be found IN.

As each fuzzy set $\hat{D}$ in case when it is FN is quite described by boundary points of reserved intervals D_λ, (λ ∈ [0,1]) , then “clear equivalent” of the equation (3) or inequality (4) will be in form

$$\alpha_\lambda x_\lambda + \beta_\lambda = \gamma_\lambda \qquad (5)$$

Or

$$\alpha_\lambda x_\lambda + \beta_\lambda \le \gamma_\lambda \qquad (6)$$

In other words the equation (3) (inequality (4)) has solution in FN then and only then when equation (5) (inequality (6)) has a solution in IN by all λ ∈ [0,1].

Further, in this work the concrete problems of identification which are solved by means of statements of theorem 1 and theorem 2 are given.

Research methods

Decision making under uncertainty is very diverse, and its complexity is much superior to similar problems in the deterministic (i.e. in the absence of uncertainty) case.

To formalize most tasks theory of decision making, under conditions of stochastic uncertainty, typically uses probability theory and based on it statistical decision theory and queuing theory.

The successful application of mathematical methods for the analysis of many applications with uncertain parameters can be performed using the methods of interval analysis.

In management decision-maker is often faced with a lot of cases when it isn’t possible to avoid the problem of the uncertainty caused by lack of clarity (fuzzy) goals and (or) restrictions.

A sure step in the formalization and analysis of such decision-making problems (as well as the application of information technology in the "non-traditional" or "humanitarian" fields, such as economics, medicine, sociology), and in building mathematical, environmental, economic, and so etc. models of specific processes, apparatus of fuzzy set theory is considered a fairly new area of applied mathematics, associated with the name of a prominent mathematician of L.A.Zadeh.

Typically, the main goal of any business is profit. In the case of construction or operation of any financial entity there is a problem of its profitability, because if its yield is below the average interest rate then its existence is meaningless in terms of profit. In financial entities (such as banks, investment funds, insurance companies, brokerage, dealer firms, etc.) the basic moments are the income from the placement and the costs in the form of payments on borrowed funds.

The most important task of commercial banks is also getting profit. For this purpose they use a variety of features, including expanding credit operations, increase services to the population. However, it is important to maintain the liquidity of each bank, which usually refers to the ability of the bank promptly and fully repay its obligations to the customers, other banks, etc.

The combination of the desire to increase profits and liquidity support should be an important landmark in the banks. However, this is not consistently enforced.

For better grounding decision-making to attract and place money it is suggested to consider the general methodology for calculating key performance indicators of the bank and their prediction. Key indicators are derived from the main purpose of the bank - attraction and allocation of funds. The main indicator for the raised funds is the average rate of interest on borrowed funds, the main indicator for allocated funds is the yield of active operations (calculated as a percentage).

Naturally, for the calculation of the indicators it is necessary to have information collected during the period. In banking, today it is not difficult, because all banks have automated systems to ensure the operations of the bank which accumulate information from the inception of the bank (the system of the bank) Thus with the availability of data there are no difficulties with their processing. However, as noted above, the data (or part of the data) are usually unclear, since they are mainly determined by the subjective (expert) way.

In the case of fuzzy methods, for example, in the financial business, as opposed to existing planning and management techniques, it is possible to use different views of the active persons engaged in the planning and decision-makers.

Analysis or Discussion

The problem of studying the relationship of economic indicators is one of the major problems of economic analysis. Therefore, any management activity is to regulate the economic variables, and it should be based on knowledge of how these variables affect other variables that are crucial to the decision-making policy. Thus, in a market economy it is impossible to directly regulate the rate of inflation, but it can be influenced by means of fiscal and monetary policy. Therefore, in particular, the relationship between the money supply and the price level should be studied.

This work is dedicated to discussion of the basic principles of modeling with fuzzy uncertainty. For example, in determining the coefficients Ai of the corresponding regression linear model (Xi are indicators of the object), the coefficients-parameters of the model are naturally identified with the fuzzy sets (in most cases - fuzzy numbers), and the simulation should be performed for the fuzzy phenomena and systems:

$$Y = A_1 * X_1 + \ldots + A_n * X_n.$$

The solution is obtained in a fuzzy form corresponding to fuzzy set information.

Note that the investment in the real economy the banks and other investors should reasonably consider not only the investment program, but also the financial, industrial, economic and socioeconomic activities of the company. Therefore, the decision-maker (DM) is interested in the study of the correlations of investments with other areas, and, above all, with the financing and production.

The ways of making investment and financial program solutions in definite situations (i.e. when the future income and expenses associated with the implementation of the project are assumed to be known) can be combined into a group of models to determine [4]:

- optimal investment program for a given for an individual investment object of a production program with a given production budget
- simultaneously both an investment and financial programs for a given production program for an individual investment property;
- simultaneously optimal investment and financial program for the given financial resources and with the involvement of the various alternatives to the model of finance. However, the transition to market-based economic relations leads to a significant expansion of investment activities through the

creation and development along with the goods and services markets of the capital market, which is a collection of various financial markets. Therefore, for a large part of investment projects future revenues and costs associated with the implementation of the project, cannot be determined unambiguously, and investors in their decisions are often faced with the uncertainty of their evaluation. The reasons for this are due to the circumstances of both the essence of the market economy (in which the future performance of an investment or other business activities depend strongly on market conditions experiencing the influence of many factors that do not depend on the efforts of investors), and the fact that the economic phenomena and processes, as a rule, are exposed to a sufficient number of non-economic factors (climatic and environmental conditions, political, social, etc.), which cannot always be an accurate assessment and prediction.

At present fuzzy logic theory has become very popular for practical applications in many fields of science. In the area of decision-making on the basis of this theory a wide range of different methods has developed. In particular, for forecasting and other planning problems in the business on the basis of data received from the experts, it is necessary to build fuzzy regressive nonlinear model. In this case, it is appropriate to use fuzzy sets as the undetermined coefficients of the model.

Among the areas of wide application of fuzzy set theory as a special place is occupied by problem of mathematical programming with fuzzy parameter values and (or) restrictions. Finally, we consider an optimization-management model combining production program with a finite number of production structures with the environmental factor (i.e. the part of output is spent on environmental protection) [9]:

$$(C, Y) \quad \max, \rightarrow$$
$$(Q, Y) \leq R,$$
$$Y \geq 0.$$

In this model:

Y – a vector of environmental program options;

C – a vector of efficiency options;

Q – a matrix of unit costs of the program versions;

R – a vector of limit for environmental costs.

With advance planning it is possible that components of the vectors C, Q and R are appointed by the coordinating center and some deviations from the "policy" value are assumed. As a result, the values of the components of these vectors parametrically depend on the degree of permissibility.

Obviously, with such an interpretation the above components of the vectors will be fuzzy sets of allowable values for each efficiency variant, unit costs and limit for environmental costs. The resulting fuzzy linear programming problem can be solved by means of the theory of fuzzy sets.

In the majority of such problems fuzzy guaranteed result is better in terms of optimization than normal (after the usual sets are subsets of the corresponding fuzzy sets).

Conclusion

In our opinion, a difficult challenge today is to choose a suitable method or software to support different processes of decision-making. Therefore, it is especially important to conduct a comparative analysis (under the condition that there is uncertainty of different kinds) of specific methods and recommendations for their use.

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ON A SEMI - MARKOVIAN STOCHASTIC PROCESS WITH FUZZY GAMMA DISTRIBUTED INTEFERENCE OF CHANCE

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In this study, a fuzzy semi - Makovian stochastic process which discribe a so called fuzzy-probabilistic inventory control model is considered. The fuzzy ergodic distribution of this process is obtained, when the amount of demand has exponential distribution and interference of chance has a gamma with fuzzy parameters.

1. Introduction and primarily discussions

The studies of renewal reward processes have an important role in the field of stochastic process, because they have been used as a model in many applications, such as; stock control, queuing, reliability and so on. There are many studies on renewal reward processes in the literature. For example, in study Brown and Solomon (1975) was obtained second order approximation for the variance of renewal reward process. Csenki (2000) considered renewal reward process with retrospective reward structure and found asymptotic expansions for the expected value. Levy and Taqqu (2000) also considered the renewal process with stable inter-renewal time intervals and stable rewards. Aliyev and Khaniyev (2014), Khaniyev et. al. (2013) was studied renewal reward process with discrete interference of chance.

Let $\{\xi_n\},\{\eta_n\},\{\theta_n\}$ and $\{\zeta_n\}, n \geq 1$ - are independent sequences of random variables defined on probability space $(\Omega, \Im, P)$, such that variables in each sequence independent and identically distributed. Suppose that ξ_n, η_n, θ_n and ζ_n take only positive values and these distribution functions be denoted by

$$\Phi(t) = P\{\xi_1 \leq t\}, t > 0, F(x) = P\{\eta_1 \leq x\}, x > 0,$$

$$\mathrm{H}(u) = P\{\theta_1 \leq u\}, u > 0, G(x, \lambda, \beta) = P\{\zeta_1 \leq z\}, z > 0,$$

Define independent renewal sequence $\{T_n\}$ and $\{Y_n\}$ as follows using the initial sequences of the random variables ξ_n and η_n as:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad Y_n = \sum_{i=1}^{n} \eta_i, n = 1,2, \dots; T_0 = Y_0 = 0.$$

Define also sequence of integer valued random variables:

$$N_0 = 0\ ;\ N_1 = N(z) = \min\{k \geq 1 : z - Y_k < 0\},$$

$$N_n \equiv N_n(\zeta_{n-1}) = \min\{k \geq N_{n-1} + 1 : \zeta_{n-1} - \left(Y_k - Y_{N_{n-1}}\right) < 0\}, n = 2,3, \dots\ .$$

Let the random variables τ_n represents the n th time of the process drops below the level 0 and γ_n represents the n th moment exit from the level 0:

$$\tau_0 = 0, \tau_1 = T_{N_1}, \gamma_0 = 0, \gamma_1 = \tau_1 + \theta_1,$$

$$\dots$$

$$\tau_n = \gamma_{n-1} + T_{N_n} - T_{N_{n-1}}, \quad \gamma_n = \tau_n + \theta_n, n = 1,2, \dots$$

Define also the counting process $\nu(t)$ which describe the number of jumps of the process $X(t)$ in the interval [0, t]:

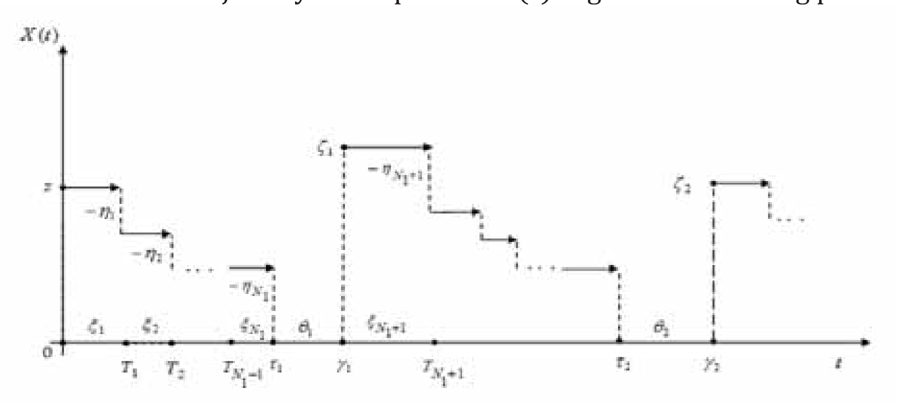
$$\nu(t) = \max\{k \geq 0 : T_k \leq t\},$$

Thus the following stochastic process can be constructed using these notations:

$$X(t) = \max\{0, \zeta_n - Y_{\nu(t)} + Y_{N_n}\}, \gamma_n \leq t < \gamma_{n+1}, n = 0,1,2, \dots ,$$

here $X(0) \equiv \zeta_0 = z$.

One of the trajectory of the process $X(t)$ is given in following picture:



The stochastic process $X(t)$ describes the so-called fuzzy-probabilistic control model. Note that in the literature there are numerous works devoted to the classical case. Despite the large number of these works, the study of this model continues in connection with the requirements of the theories of inventory management, queuing, reliability, insurance, etc.

In this model there are important random variables:

η_n - demand quantity,

ξ_n- time between sequential demands,

θ_n- time of the process stay at the level 0,

ζ_n- the size of the jumps of the process $X(t)$ after it hits the level 0.

We assume that, random variables ξ_n can have a kind of different distribution an exponential distribution, η_n and ζ_n have the some distribution with fuzzy parameters. Consequently, the process $X(t)$ we called a fuzzy semi-Markovian process. Note that, the idea of considering fuzzy-probabilistic models belongs to professor Zadeh (see, for example, Zadeh (1968), Zadeh (2011)).

For the $\tilde{P}[c,d]$ - fuzzy probability of obtaining a value from the interval [c,d] considered in such a distribution (see, for example, Buckley and Eslami (2004)):

$$\tilde{P}[c,d][\alpha] = [p_1(\alpha), p_2(\alpha)],$$

where

$$p_1(\alpha) = min\left\{\int_c^d g(x,\beta,\lambda)dx \,\middle|, \beta \in \tilde{\beta}[\alpha], \lambda \in \tilde{\lambda}[\alpha]\right\},$$

$$p_2(\alpha) = max\left\{\int_c^d g(x,\beta,\lambda)dx \,\middle|, \beta \in \tilde{\beta}[\alpha], \lambda \in \tilde{\lambda}[\alpha]\right\}.$$

and $g(x,\beta,\lambda) = \frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x}, x > 0$ is density function of classical gamma distribution with parameters $\beta, \lambda > 0$.

Suppose that the mathematical expectation in the fuzzy distribution under consideration is $\tilde{E}$. Then its α-cuts are

$$\tilde{E}[\alpha] = \{\int_0^\infty x g(x,\beta,\lambda)dx \,|, \beta \in \tilde{\beta}[\alpha], \lambda \in \tilde{\lambda}[\alpha]\}.$$

On the other hand, suppose that α-cut of the variance of fuzzy distribution define as follows:

$$\tilde{D}[\alpha] = \{\int_0^\infty (x-E)^2 g(x,\beta,\lambda)dx \,| \beta \in \tilde{\beta}[\alpha], \lambda \in \tilde{\lambda}[\alpha], E \in \tilde{E}[\alpha]\}.$$

By calculating the integrals of both sets, we can find a fuzzy mathematical expectation and a fuzzy variance in this fuzzy distribution.From there we can easily find

$$\tilde{E} = \frac{\tilde{\beta}}{\tilde{\lambda}} \text{ and } \tilde{D} = \frac{\tilde{\beta}}{\tilde{\lambda}^2}.$$

2. Fuzzy ergodic distribution of the process

In this section we will give exact formulas for the ergodic distribution of the investigated process $X(t)$. Let us denote ergodic distribution of the process $X(t)$ as $Q_X(x) \equiv \lim_{t\to\infty} P\{X(t) \le x\}$. Fuzzy ergodic distribution function we denote as $\tilde{Q}_X(x, \mu, \tilde{\beta}, \tilde{\lambda})$.

Theorem 1. *Let initial sequence $\{\xi_n\}, \{\eta_n\}, \{\theta_n\}$ and $\{\zeta_n\}, n \ge 1$ – satisfies the following supplementary conditions:*

1) $E\xi_1 < \infty$;

2)$E\theta_1 < \infty$;

3) random variable η_1 has the exponential distribution with parameter $\mu > 0$;

4) random variables $\{\zeta_n\}, n \ge 1$ has the gamma distribution with fuzzy parameters $(\tilde{\beta}, \tilde{\lambda})$.

Then the process $X(t)$ is ergodic and ergodic distribution function has the following explicit form:

$$\tilde{Q}_X(x,\mu,\tilde{\beta},\tilde{\lambda}) = 1 - \frac{\mu x}{\tilde{\lambda} + \mu\tilde{\beta} + \tilde{\lambda}K}\tilde{g}(x,\tilde{\beta},\tilde{\lambda}) + \frac{\tilde{\lambda} + \tilde{\mu}\tilde{\beta} - \mu\tilde{\lambda}x}{\tilde{\lambda} + \mu\tilde{\beta} + \tilde{\lambda}K}\left(1 - \tilde{G}(x,\tilde{\beta},\tilde{\lambda})\right),$$

where

$$\tilde{g}(x,\tilde{\beta},\tilde{\lambda}) = \frac{\tilde{\lambda}^{\tilde{\beta}}}{\Gamma(\tilde{\beta})}x^{\tilde{\beta}-1}e^{-\tilde{\lambda}x},$$

$$\tilde{G}(x,\tilde{\beta},\tilde{\lambda}) = \frac{\tilde{\lambda}^{\tilde{\beta}}}{\Gamma(\tilde{\beta})}\int_0^x t^{\tilde{\beta}-1}e^{-\tilde{\lambda}x}\,dt.$$

Proof. Considered process belongs to a wide class of the processes which is called as “Processes with a discrete interference of chance” in literature. For this class, the general ergodic theorem is given in monograph Gihman and Skorohod (1975). Conditions 1)-4) of this theorem provide the fulfillment of the conditions of the general ergodic theorem.

Now we can find α-cut of the $\tilde{Q}_X(x,\mu,\tilde{\beta},\tilde{\lambda})$.

Theorem 2. Let conditions of Theorem 1 be satisfied. In the distribution function $\tilde{Q}_X(x,\mu,\tilde{\beta},\tilde{\lambda})$, when the parameters $\tilde{\beta}$ and $\tilde{\lambda}$ are a fuzzy triangular numbers $\tilde{\beta} = (a_1/a_2/a_3)$, $a_i > 0$, $i = \overline{1,3}$ and $\tilde{\lambda} = (b_1/b_2/b_3), b_i > 0$, $i = \overline{1,3}$. Then the α-cut for each x of the fuzzy distribution $\tilde{Q}_X(x,\mu,\tilde{\beta},\tilde{\lambda})$ is

$$\tilde{Q}_X(x,\mu,\tilde{\beta},\tilde{\lambda})[\alpha] = [Q_1^\alpha(x), Q_2^\alpha(x)],$$

where

$$Q_1^\alpha(x) = min\{Q_X(x,\mu,\beta,\lambda) | \beta \in \beta[\alpha], \lambda \in \lambda[\alpha]\};$$

$$Q_2^\alpha(x) = max\{Q_X(x,\mu,\beta,\lambda) | \beta \in \beta[\alpha], \lambda \in \lambda[\alpha]\}.$$

Proof. Let first we introduce the triangular fuzzy numbers $\tilde{\beta} = (a_1/a_2/a_3)$, $a_i > 0, i = \overline{1,3}$ and $\tilde{\lambda} = (b_1/b_2/b_3), b_i > 0, i = \overline{1,3}$ with the following membership functions:

$$\mu_{\tilde{\beta}}(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & x > a_3 \end{cases} \quad \text{and} \quad \mu_{\tilde{\lambda}}(x) = \begin{cases} 0, & x < b_1 \\ \frac{x-b_1}{b_2-b_1}, & b_1 \le x \le b_2 \\ \frac{b_3-x}{b_3-b_2}, & b_2 \le x \le b_3 \\ 0, & x > b_3 \end{cases}$$

Assuming that the number $\mu > 0$ remain some number, we will additionally assume that β and λ is triangular fuzzy numbers $\tilde{\beta} > 0$ and $\tilde{\lambda} > 0$, which defined above. Then their α-cuts are $\tilde{\beta}[\alpha] = [\beta_1(\alpha), \beta_2(\alpha)]$ and $\tilde{\lambda}[\alpha] = [\lambda_1(\alpha), \lambda_2(\alpha)]$,

where

$$\begin{cases} \beta_1(\alpha) = a_1 + (a_2 - a_1)\alpha \\ \beta_2(\alpha) = a_3 - (a_3 - a_2)\alpha \end{cases}$$

and

$$\begin{cases} \lambda_1(\alpha) = b_1 + (b_2 - b_1)\alpha \\ \lambda_2(\alpha) = b_3 - (b_3 - b_2)\alpha \end{cases}.$$

Second we want find the α-cut of $\tilde{Q}_X(x, \mu, \tilde{\beta}, \tilde{\lambda})$ for every x. From Corollary 1 we can see the α-cut of $\tilde{Q}_X(x, \mu, \tilde{\beta}, \tilde{\lambda})$ is

$$\tilde{Q}_X(x, \mu, \tilde{\beta}, \tilde{\lambda})\,[\alpha] =$$

$$= \left\{ \frac{\lambda + \beta\mu + \lambda K - \mu x g_{\beta,\lambda}(x) + (\lambda + \beta\mu - \lambda\mu x)\left(1 - G_{\beta,\lambda}(x)\right)}{\lambda + \beta\mu + \lambda K} \middle| \beta \in \beta[\alpha], \lambda \in \lambda[\alpha] \right\}$$

In other word

$$\tilde{Q}_X(x, \mu, \tilde{\beta}, \tilde{\lambda})[\alpha] = \{Q_X(x, \mu, \beta, \lambda) | \beta \in \beta[\alpha], \lambda \in \lambda[\alpha]\}.$$

If we denote

$$\tilde{Q}_X(x, \mu, \tilde{\beta}, \tilde{\lambda})[\alpha] = [Q_1^\alpha(x), Q_2^\alpha(x)],$$

then

$$Q_1^\alpha(x) = min\{Q_X(x, \mu, \beta, \lambda) | \beta \in \beta[\alpha], \lambda \in \lambda[\alpha]\};$$

$$Q_2^\alpha(x) = max\{Q_X(x, \mu, \beta, \lambda) | \beta \in \beta[\alpha], \lambda \in \lambda[\alpha]\}.$$

Theorem 3. Let conditions of Theorem 2 be satisfied. Then for each x the fuzzy distribution $\tilde{Q}_X(x, \mu, \tilde{\beta}, \tilde{\lambda})$ has the following approximate formula:

$$\tilde{Q}_X(x, \mu, \tilde{\beta}, \tilde{\lambda}) \cong \left(Q_1(x)/Q_2(x)/Q_3(x)\right),$$

where

$$Q_1(x) = min\{Q_X(x, \mu, \beta, \lambda) | \beta \in [a_1, a_3], \lambda \in [b_1, b_3]\},$$

$$Q_2(x) = Q_X(x, \mu, a_2, b_2),$$

$$Q_3(x) = max\{Q_X(x, \mu, \beta, \lambda) | \beta \in [a_1, a_3], \lambda \in [b_1, b_3]\}.$$

Proof. From Theorem 2, we have

$$\tilde{Q}_X(x,\mu,\tilde{\beta},\tilde{\lambda})[0] = [Q_1^0(x), Q_2^0(x)] =$$

$$= [min\{Q_X(x,\mu,\beta,\lambda)|\beta \in \beta[0], \lambda \in \lambda[0]\}, max\{Q_X(x,\mu,\beta,\lambda)|\beta \in \beta[0], \lambda \in \lambda[0]\}]$$

$$= [min\{Q_X(x,\mu,\beta,\lambda)|\beta \in [a_1,a_3], \lambda \in [b_1,b_3]\}, max\{Q_X(x,\mu,\beta,\lambda)|\beta \in [a_1,a_3], \lambda \in [b_1,b_3]\}]$$

and

$$\tilde{Q}_X(x,\mu,\tilde{\beta},\tilde{\lambda})[1] = [Q_1^1(x), Q_2^1(x)] =$$

$$= [min\{Q_X(x,\mu,\beta,\lambda)|\beta \in \beta[1], \lambda \in \lambda[1]\}, max\{Q_X(x,\mu,\beta,\lambda)|\beta \in \beta[1], \lambda \in \lambda[1]\}]$$

$$= [Q_X(x,\mu,a_2,b_2), Q_X(x,\mu,a_2,b_2)] = Q_X(x,\mu,a_2,b_2).$$

Then approximately,

$$\tilde{Q}_X(x,\mu,\tilde{\beta},\tilde{\lambda}) \cong \left(Q_1(x)/Q_2(x)/Q_3(x)\right),$$

where

$$Q_1(x) = min\{Q_X(x,\mu,\beta,\lambda)|\beta \in [a_1,a_3], \lambda \in [b_1,b_3]\},$$

$$Q_2(x) = Q_X(x,\mu,a_2,b_2),$$

$$Q_3(x) = max\{Q_X(x,\mu,\beta,\lambda)|\beta \in [a_1,a_3], \lambda \in [b_1,b_3]\}.$$

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FUNDAMENTAL SOLUTION OF A THIRD ORDER THREE-DIMENSIONAL COMPOSITE EQUATION

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The paper is devoted to finding a fundamental solution, basic relationships and necessary conditions to a boundary value problem for a three-dimensional third order composite equation with nonlocal boundary value conditions.

Keywords: three-dimensional, third-order composite equation, fundamental solution, basic relationships, necessary conditions.

Introduction

As is known, for an ordinary differential equation the number of additional conditions (Cauchy conditions or boundary conditions) always coincides with the order of the equation in question [1].

The non-local boundary conditions free us from the above misunderstanding between ordinary differential equations and partial differential equations. The authors, precisely for nonlocal boundary value problems, found the possibility of proving Fredholm property with the help of the derivation of the so-called necessary conditions and their regularization.

The idea of necessary conditions was first used by A.V. Bitsadze for the Laplace equation [2, p.185]. We are going to obtain the necessary conditions from the basic relationships which are deduced only from the equation under consideration and are analogs of the Lagrange formula for an ordinary differential equations and for partial differential equations they are obtained from the second Green formula. Necessary conditions show to which relationships the required solution and its partial derivatives must satisfy on the boundary of the domain and that we cannot arbitrarily define the desired function and its partial derivatives on the boundary of the domain even in the class of admissible functions (continuous or continuously differentiable). That is, if the boundary conditions do not satisfy the necessary conditions then the stated boundary value problem has no solution in this class. If the boundary conditions are a linear combination of these necessary conditions then the solution will not be unique and the boundary value problem will have infinitude of solutions.

If the necessary conditions are singular but they can be regularized with the help of the very non-local boundary conditions and then used to prove the Fredholm property of the stated boundary value problem by an original scheme [3], [4].

Considering the nonlocal boundary conditions for a partial differential equation we get the opportunity to investigate the solution of boundary value problems for differential equations of both even and odd order.

1. Statement of the problem

We are going to consider a three-dimensional composite equation

$$L_1 u_1(x) = \frac{\partial^3 u_1(x)}{\partial x_3^2 \partial x_2} + \frac{\partial}{\partial x_2}\left(\frac{\partial^2 u_1(x)}{\partial x_1^2} + \frac{\partial^2 u_1(x)}{\partial x_2^2}\right) = 0\,,\ x \in D_1, \tag{1.1}$$

$$x = (x_1, x_2, x_3) \in D_1,$$

in three-dimensional domain $D = \{x = (x_1, x_2, x_3) \in R^3\}$ with Lyapunov boundary Γ convex in direction x_3, with nonlocal boundary conditions:

$$l_i u = \sum_{m=0}^{1}\sum_{k=1}^{2}\left[\alpha_{ik,11}^{(m)}(x')\frac{\partial^2 u_k(x)}{\partial x_1^2}\Big|_{x_3=\gamma_m(x')} + \alpha_{ik,22}^{(m)}(x')\frac{\partial^2 u_k(x)}{\partial x_2^2}\Big|_{x_3=\gamma_m(x')} + \alpha_{ik,33}^{(m)}(x')\frac{\partial^2 u_k(x)}{\partial x_3^2}\Big|_{x_3=\gamma_m(x')} + \right.$$

$$\left. + \alpha_{ik,12}^{(m)}(x')\frac{\partial^2 u_k(x)}{\partial x_1 \partial x_2}\Big|_{x_3=\gamma_m(x')} + \alpha_{ik,13}^{(m)}(x')\frac{\partial^2 u_k(x)}{\partial x_1 \partial x_3}\Big|_{x_3=\gamma_m(x')} + \alpha_{ik,23}^{(m)}(x')\frac{\partial^2 u_k(x)}{\partial x_2 \partial x_3}\Big|_{x_3=\gamma_m(x')}\right] +$$

$$+ \sum_{j=1}^{3}\left[\alpha_{ij1}^{(1)}(x')\frac{\partial u_1(x)}{\partial x_j}\Big|_{x_3=\gamma_1(x')} + \alpha_{ij1}^{(0)}(x')\frac{\partial u_1(x)}{\partial x_j}\Big|_{x_3=\gamma_0(x')} + \right.$$

$$\left. + \alpha_{ij2}^{(2)}(x')\frac{\partial u_2(x)}{\partial x_j}\Big|_{x_3=\gamma_2(x')} + \alpha_{ij2}^{(0)}(x')\frac{\partial u_2(x)}{\partial x_j}\Big|_{x_3=\gamma_0(x')}\right] +$$

$$+ \alpha_{i1}^{(1)}(x')u_1(x', \gamma_1(x')) + \alpha_{i1}^{(0)}(x')u_1(x', \gamma_0(x')) +$$

$$+ \alpha_{i2}^{(2)}(x')u_2(x', \gamma_2(x')) + \alpha_{i2}^{(0)}(x')u_2(x', \gamma_0(x')) = f_i(x'), \tag{1.2}$$

$$u(x) = \begin{cases} u_1(x),\ x \in D_1, x_3 > 0, \\ u_2(x),\ x \in D_2, x_3 < 0, \end{cases}$$

$$u(x) = f_0(x), \quad x \in L = \overline{\Gamma}_1 \cap \overline{\Gamma}_2\,, \tag{1.3}$$

where domain $S \subset Ox_1x_2$ is the projection of D onto plane $Ox_1x_2 = Ox'$, Γ_1 and Γ_2 are the upper and the lower half-surfaces of the boundary Γ respectively defined as $\Gamma_k = \{\xi = (\xi_1, \xi_2, \xi_3) : \xi_3 = \gamma_k(\xi'),\ \xi' = (\xi_1, \xi_2) \in S\}$, where $\xi_3 = \gamma_k(\xi_1, \xi_2), k = 1, 2$, are the equations of Γ_1 and Γ_2 respectively, functions $\gamma_k(\xi'), k = 1, 2$, are twice-differentiable in *S*; *L* is the equator connecting Γ_1 and Γ_2: $L = \overline{\Gamma}_1 \cap \overline{\Gamma}_2$.

We denote the projection of domain D onto the plane $Ox_1x_2 = Ox'$ as $\Gamma_0 = \{\xi = (\xi_1, \xi_2, \xi_3) : \xi_3 = \gamma_0(\xi') = 0,\ \xi' = (\xi_1, \xi_2) \in S\}$. The coefficients $\alpha_{ijp}^{(k)}(x')$, $\alpha_{ip}^{(k)}(x')$, $i, p = 1,2;\ j = 1,2,3; k = 0,1$, satisfy Hölder condition in *S*, the right-hand sides $f_i(x'),\ i = 1,2$, and $f_0(x)$ are continuous in S and *L* respectively.

The linear independent boundary conditions (1.2) as if «saw» the values of the desired function $u(x)$ and its partial derivatives on the half-surfaces Γ_1, Γ_2 and Γ_0.

2. Fundamental solution

The f**undamental solution** of three-dimensional equation (1.1) has been obtained by means of Fourier transformation of equation (1.1) in the form [1]:

$$U(x-\xi)=\frac{i}{(2\pi)^3}\int_{R^3}\frac{e^{i(\alpha,x-\xi)}}{\alpha_2\left(\alpha_1^2+\alpha_2^2+\alpha_3^2\right)}d\alpha. \qquad (2.1)$$

The direct integration of super-singular integral (2.1) can be done by Hörmander ladder [5]. But the authors found another way: to represent the equation (1.1) in the form:

$$Lu(x)=\frac{\partial}{\partial x_2}\left(\frac{\partial^2 u(x)}{\partial x_1^2}+\frac{\partial^2 u(x)}{\partial x_2^2}+\frac{\partial^2 u(x)}{\partial x_3^2}\right)=0,$$

or

$$Lu(x)=\Delta\frac{\partial}{\partial x_2}u(x)=0. \qquad (2.2)$$

Then the fundamental solution $U(x)$ of equation (2.2) satisfies the equation:

$$LU(x)=\Delta\frac{\partial}{\partial x_2}U(x)=\delta(x).$$

As the fundamental solution of three-dimensional Laplace equation is

$$\frac{\partial}{\partial x_2}U(x)=-\frac{1}{4\pi|x|},$$

then

$$U(x)=-\frac{1}{4\pi}\int\frac{dt}{\sqrt{x_1^2+x_3^2+t^2}}. \qquad (2.3)$$

Let us calculate the integral in the right-hand side of (2.3). Taking into account that

$$\int\frac{1}{\sqrt{x_1^2+x_3^2+t^2}}dt=\ln\left|t+\sqrt{x_1^2+x_3^2+t^2}\right|+C$$

we obtain the fundamental solution of equation (1.1):

$$U(x)=-\frac{1}{4\pi}\ln\left|x_2+\sqrt{x_1^2+x_2^2+x_3^2}\right|+C. \qquad (2.4)$$

3. Basic relationships and necessary conditions

Multiplying (1.1) by fundamental solution (2.4), integrating over the domain *D* and taki8ng into account $L_xU(x-\xi)=\delta(x-\xi)$ where $\delta(x-\xi)$

is Dirac’s δ -function we obtain **the first basic relationship**:

$$\int_{\Gamma_1} \Delta u_1(x) U_1(x-\xi)\cos(\nu, x_2)dx - \int_{\Gamma_1} \frac{\partial u_1(x)}{\partial x_3}\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu, x_3)dx +$$

$$-\int_{\Gamma_1}\left[\frac{\partial u_1(x)}{\partial x_1}\cos(\nu, x_1) + \frac{\partial u_1(x)}{\partial x_2}\cos(\nu, x_2) + \frac{\partial u_1(x)}{\partial x_3}\cos(\nu, x_3)\right]\frac{\partial U_1(x-\xi)}{\partial x_2}dx +$$

$$+\int_{\Gamma_1} u_1(x)\left[\frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_1}\cos(\nu, x_1) + \frac{\partial^2 U_1(x-\xi)}{\partial x_2^2}\cos(\nu, x_2) + \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_3}\cos(\nu, x_3)\right]dx =$$

$$= \int_{D_1} u_1(x)\delta(x-\xi)dx = \begin{cases} u_1(\xi), & \xi \in D_1, \\ \frac{1}{2}u_1(\xi), & \xi \in \Gamma_1. \end{cases}$$

or

$$\int_{\Gamma_1} \Delta u_1(x) U_1(x-\xi)\cos(\nu, x_2)dx -$$

$$-\int_{\Gamma_1} \frac{\partial u_1(x)}{\partial \nu}\frac{\partial U_1(x-\xi)}{\partial x_2}dx + \int_{\Gamma_1} u_1(x)\frac{\partial}{\partial x_2}\frac{\partial U_1(x-\xi)}{\partial \nu}dx =$$

$$= \int_{D_1} u_1(x)\delta(x-\xi)dx = \begin{cases} u_1(\xi), & \xi \in D_1, \\ \frac{1}{2}u_1(\xi), & \xi \in \Gamma_1. \end{cases} \qquad (3.1)$$

The first relationship in (3.1) gives the representation of a general solution of (1.1) and the second one is **the first necessary condition**:

$$\frac{1}{2}u_1(\xi) = \int_{\Gamma_1} \Delta u_1(x) U_1(x-\xi)\cos(\nu, x_2)dx -$$

$$-\int_{\Gamma_1} \frac{\partial u_1(x)}{\partial \nu}\frac{\partial U_1(x-\xi)}{\partial x_2}dx + \int_{\Gamma_1} u_1(x)\frac{\partial}{\partial x_2}\frac{\partial U_1(x-\xi)}{\partial \nu}dx. \qquad (3.2)$$

Let us calculate first-order partial derivatives of the fundamental solution:

$$\frac{\partial U_1(x-\xi)}{\partial x_1} = \frac{\partial}{\partial x_1}\left(-\frac{1}{4\pi}\ln\left|x_2 + |x|\right|\right) = -\frac{1}{4\pi}\frac{x_1}{\left(x_2 + |x|\right)|x|},$$

$$\frac{\partial U_1(x-\xi)}{\partial x_2} = \frac{\partial}{\partial x_2}\left(-\frac{1}{4\pi}\ln\left|x_2 + |x|\right|\right) = -\frac{1}{4\pi}\frac{x_2 + |x|}{\left(x_2 + |x|\right)|x|},$$

$$\frac{\partial U_1(x-\xi)}{\partial x_3} = \frac{\partial}{\partial x_3}\left(-\frac{1}{4\pi}\ln\left|x_2 + |x|\right|\right) = -\frac{1}{4\pi}\frac{x_3}{\left(x_2 + |x|\right)|x|}. \qquad (3.3)$$

Substituting (3.3) into (3.2) we get **the first necessary condition** in the form

$$\frac{1}{2}u_1(\xi)=\int_{\Gamma_1}\Delta u_1(x)\left(-\frac{1}{4\pi}\ln\left|x_2+\sqrt{x_1^2+x_2^2+x_3^2}\right|\right)\cos(\nu,x_2)dx-$$

$$-\int_{\Gamma_1}\frac{\partial u_1(x)}{\partial\nu}\left(-\frac{1}{4\pi}\frac{x_2+|x|}{(x_2+|x|)|x|}\right)dx+\int_{\Gamma_1}u_1(x)\frac{\partial}{\partial x_2}\frac{\partial U_1(x-\xi)}{\partial\nu}dx\ ,\ \xi\in\Gamma\ . \quad (3.4)$$

So, we have proved the theorem:

Theorem 3.1. *Let domain* $D\subset R^3$ *be convex along axis* Ox_3 *and bounded with piece-wise Lyapunov boundary* Γ. *Then the first necessary condition* (3.4) *is singular*.

To derive **the second basic relationships** we multiply (1.1) by $\frac{\partial U_1(x-\xi)}{\partial x_i}$, $i=\overline{1,3}$, and integrate it over D_1 applying integration by parts:

$$\int_{D_1}Lu_1(x)\frac{\partial^2U_1(x-\xi)}{\partial x_i}dx=\int_{\Gamma'}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_i}\cos(\nu,x_2)dx-\int_{D_1}\Delta u_1(x)\frac{\partial^2U_1(x-\xi)}{\partial x_2\partial x_i}dx=$$

$$=\int_{\Gamma'}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_i}\cos(\nu,x_2)dx-\sum_{k=1}^{3}\int_{D_1}\frac{\partial^2u_1(x)}{\partial x_k^2}\frac{\partial^2U_1(x-\xi)}{\partial x_2\partial x_i}dx\,. \quad (3.5)$$

As

$$\int_{D_1}\frac{\partial^2u_1(x)}{\partial x_k^2}\frac{\partial^2U_1(x-\xi)}{\partial x_2\partial x_i}dx=\int_{\Gamma_1}\frac{\partial^2u_1(x)}{\partial x_k^2}\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_i)dx-$$

$$-\int_{D_1}\frac{\partial^3u_1(x)}{\partial x_i\partial x_k^2}\frac{\partial U_1(x-\xi)}{\partial x_2}dx=\int_{\Gamma_1}\frac{\partial^2u_1(x)}{\partial x_k^2}\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_i)dx-$$

$$-\int_{\Gamma_1}\frac{\partial^2u_1(x)}{\partial x_i\partial x_k}\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_k)dx+\int_{D_1}\frac{\partial^2u_1(x)}{\partial x_i\partial x_k}\frac{\partial^2U_1(x-\xi)}{\partial x_k\partial x_2}dx=$$

$$=\int_{\Gamma_1}\frac{\partial^2u_1(x)}{\partial x_k^2}\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_i)dx-\int_{\Gamma_1}\frac{\partial^2u_1(x)}{\partial x_i\partial x_k}\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_k)dx+$$

$$+\int_{\Gamma_1}\frac{\partial u_1(x)}{\partial x_i}\frac{\partial^2U_1(x-\xi)}{\partial x_k\partial x_2}\cos(\nu,x_k)dx-\int_{D_1}\frac{\partial u_1(x)}{\partial x_i}\frac{\partial^3U_1(x-\xi)}{\partial x_k^2\partial x_2}dx \quad (3.6)$$

then substituting (3.6) into (3.5) for $k=\overline{1,3}$, we obtain:

$$\int_{D_1}Lu_1(x)\frac{\partial^2U_1(x-\xi)}{\partial x_i}dx=\int_{\Gamma'}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_i}\cos(\nu,x_2)dx-$$

$$-\int_{\Gamma_1}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_i)dx+\int_{\Gamma_1}\sum_{k=1}^{3}\frac{\partial^2 u_1(x)}{\partial x_i\partial x_k}\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_k)dx-$$

$$-\int_{\Gamma_1}\sum_{k=1}^{3}\frac{\partial u_1(x)}{\partial x_i}\frac{\partial^2 U_1(x-\xi)}{\partial x_k\partial x_2}\cos(\nu,x_k)dx+\int_{D_1}\frac{\partial u_1(x)}{\partial x_i}LU_1(x-\xi)dx. \qquad (3.7)$$

Taking into account that $\sum_{k=1}^{3}\frac{\partial^2 u_1(x)}{\partial x_i\partial x_k}\cos(\nu,x_k)=\frac{\partial}{\partial \nu}\frac{\partial u_1(x)}{\partial x_i}$, we rewrite (3.7):

$$0=\int_{D_1}Lu_1(x)\frac{\partial^2 U_1(x-\xi)}{\partial x_i}dx=\int_{\Gamma'}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_i}\cos(\nu,x_2)dx-$$

$$-\int_{\Gamma_1}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_i)dx+\int_{\Gamma_1}\frac{\partial U_1(x-\xi)}{\partial x_2}\frac{\partial}{\partial \nu}\left(\frac{\partial u_1(x)}{\partial x_i}\right)dx-$$

$$-\int_{\Gamma_1}\frac{\partial u_1(x)}{\partial x_i}\frac{\partial}{\partial \nu}\left(\frac{\partial U_1(x-\xi)}{\partial x_2}\right)dx+\int_{D_1}\frac{\partial u_1(x)}{\partial x_i}LU_1(x-\xi)dx,$$

whence we obtain **the second basic relationships**

$$\int_{\Gamma_1}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_i)dx-\int_{\Gamma'}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_i}\cos(\nu,x_2)dx+$$

$$+\int_{\Gamma_1}\frac{\partial u_1(x)}{\partial x_i}\frac{\partial}{\partial \nu}\left(\frac{\partial U_1(x-\xi)}{\partial x_2}\right)dx-\int_{\Gamma_1}\frac{\partial U_1(x-\xi)}{\partial x_2}\frac{\partial}{\partial \nu}\left(\frac{\partial u_1(x)}{\partial x_i}\right)dx=$$

$$=\begin{cases}\frac{\partial u_1(\xi)}{\partial \xi_i}, & \xi\in D_1,\\ \frac{1}{2}\frac{\partial u_1(\xi)}{\partial \xi_i}, & \xi\in\Gamma_1,\end{cases}\quad i=\overline{1,3}, \qquad (3.8)$$

the latter of which are **the second necessary conditions ($\xi\in\Gamma_1$)**:

$$\frac{1}{2}\frac{\partial u_1(\xi)}{\partial \xi_i}$$

$$=\int_{\Gamma_1}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_2}\cos(\nu,x_i)dx-\int_{\Gamma'}\Delta u_1(x)\frac{\partial U_1(x-\xi)}{\partial x_i}\cos(\nu,x_2)dx+$$

$$+\int_{\Gamma_1}\frac{\partial u_1(x)}{\partial x_i}\frac{\partial}{\partial \nu}\left(\frac{\partial U_1(x-\xi)}{\partial x_2}\right)dx-\int_{\Gamma_1}\frac{\partial U_1(x-\xi)}{\partial x_2}\frac{\partial}{\partial \nu}\left(\frac{\partial u_1(x)}{\partial x_i}\right)dx,\ i=\overline{1,3}. \qquad (3.9)$$

To obtain the **third basic relationships** we multiply (1.1) by $\frac{\partial^2 U_1(x-\xi)}{\partial x_i \partial x_j}$, $i, j = \overline{1,3}$, and integrate it over D_1 we get

$$0 = \int_{D_1} Lu_1(x) \frac{\partial^2 U_1(x-\xi)}{\partial x_i \partial x_j} dx =$$

$$= \int_{\Gamma'} \Delta u_1(x) \frac{\partial^2 U_1(x-\xi)}{\partial x_i \partial x_j} \cos(\nu, x_2) dx - \int_{D_1} \Delta u_1(x) \frac{\partial^3 U_1(x-\xi)}{\partial x_2 \partial x_i \partial x_j} dx =$$

$$= \int_{\Gamma'} \Delta u_1(x) \frac{\partial^2 U_1(x-\xi)}{\partial x_i \partial x_j} \cos(\nu, x_2) dx - \sum_{k=1}^{3} \int_{D_1} \frac{\partial^2 u_1(x)}{\partial x_k^2} \frac{\partial^3 U_1(x-\xi)}{\partial x_2 \partial x_i \partial x_j} dx . \qquad (3.10)$$

As

$$\int_{D_1} \frac{\partial^2 u_1(x)}{\partial x_k^2} \frac{\partial^3 U_1(x-\xi)}{\partial x_2 \partial x_i \partial x_j} dx =$$

$$= \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_k^2} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_j) dx - \int_{D_1} \frac{\partial^3 u_1(x)}{\partial x_j \partial x_k^2} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} dx =$$

$$= \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_k^2} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_j) dx - \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_k) dx +$$

$$+ \int_{D_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^3 U_1(x-\xi)}{\partial x_k \partial x_2 \partial x_i} dx = \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_k^2} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_j) dx -$$

$$- \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_k) dx + \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_i) dx -$$

$$- \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_k) dx + \int_{D_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} \frac{\partial^3 U_1(x-\xi)}{\partial x_k^2 \partial x_2} dx \qquad (3.11)$$

then taking into account (3.11) for $k = 1,2,3$ we obtain

$$\int_{D_1} \Delta u_1(x) \frac{\partial^3 U_1(x-\xi)}{\partial x_2 \partial x_i \partial x_j} dx = \sum_{k=1}^{3} \int_{D_1} \frac{\partial^2 u_1(x)}{\partial x_k^2} \frac{\partial^3 U_1(x-\xi)}{\partial x_2 \partial x_i \partial x_j} dx = \int_{\Gamma_1} \sum_{k=1}^{3} \frac{\partial^2 u_1(x)}{\partial x_k^2} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_j) dx -$$

$$- \sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_k) dx + \sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_i) dx -$$

$$- \sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_k) dx + \sum_{k=1}^{3} \int_{D_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} \frac{\partial^3 U_1(x-\xi)}{\partial x_k^2 \partial x_2} dx$$

or

$$\int_{D_1} \Delta u_1(x) \frac{\partial^3 U_1(x-\xi)}{\partial x_2 \partial x_i \partial x_j} dx = \int_{\Gamma_1} \Delta u_1(x) \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_j) dx -$$

$$-\sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_k) dx + \sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_i) dx -$$

$$-\sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_k) dx + \sum_{k=1}^{3} \int_{D_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} \frac{\partial}{\partial x_2} \Delta U_1(x-\xi) dx \,. \quad (3.12)$$

Substituting (3.12) in (3.10), we obtain

$$0 = \int_{D_1} L u_1(x) \frac{\partial^2 U_1(x-\xi)}{\partial x_i \partial x_j} dx = \int_{\Gamma'} \Delta u_1(x) \frac{\partial^2 U_1(x-\xi)}{\partial x_i \partial x_j} \cos(\nu, x_2) dx - \int_{D_1} \Delta u_1(x) \frac{\partial^3 U_1(x-\xi)}{\partial x_2 \partial x_i \partial x_j} dx =$$

$$= \int_{\Gamma'} \Delta u_1(x) \frac{\partial^2 U_1(x-\xi)}{\partial x_i \partial x_j} \cos(\nu, x_2) dx - \int_{\Gamma_1} \Delta u_1(x) \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_j) dx +$$

$$+\sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_k) dx - \sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_i) dx +$$

$$+\sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_k) dx - \int_{D_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} L U_1(x-\xi) dx \,. \quad (3.13)$$

Taking into account in (3.13) that $LU_1(x-\xi) = \delta(x-\xi)$ we obtain **the third basic relationships** ($i, j = \overline{1,3}$)

$$\int_{\Gamma_1} \Delta u_1(x) \left(\frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_j) - \frac{\partial^2 U_1(x-\xi)}{\partial x_i \partial x_j} \cos(\nu, x_2) \right) dx +$$

$$+\sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_j \partial x_k} \left(\frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_i) - \frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_i} \cos(\nu, x_k) \right) dx -$$

$$-\sum_{k=1}^{3} \int_{\Gamma_1} \frac{\partial^2 u_1(x)}{\partial x_i \partial x_j} \frac{\partial^2 U_1(x-\xi)}{\partial x_k \partial x_2} \cos(\nu, x_k) dx = \begin{cases} \dfrac{\partial^2 u_1(x)}{\partial x_i \partial x_j}, \ \xi \in D_1, \\[2mm] \dfrac{1}{2} \dfrac{\partial^2 u_1(x)}{\partial x_i \partial x_j}, \ \xi \in \Gamma_1. \end{cases} \quad (3.14)$$

Calculating third-order derivatives

$$\frac{\partial^2 U_1(x-\xi)}{\partial x_1{}^2} = \frac{\partial}{\partial x_1} \left(-\frac{1}{4\pi} \frac{x_1}{(x_2 + |x|)|x|} \right) = -\frac{1}{4\pi} \frac{x_2(x_2^2 + x_3^2) + |x|^3 - 2x_1^2 |x|}{(x_2 + |x|)^2 |x|^3},$$

$$\frac{\partial^2 U_1(x-\xi)}{\partial x_2{}^2} = \frac{\partial}{\partial x_2}\left(-\frac{1}{4\pi}\frac{x_2+|x|}{(x_2+|x|)|x|}\right) = \frac{1}{4\pi}\frac{x_2}{|x|^3},$$

$$\frac{\partial^2 U_1(x-\xi)}{\partial x_3{}^2} = \frac{\partial}{\partial x_3}\left(-\frac{1}{4\pi}\frac{x_3}{(x_2+|x|)|x|}\right) = -\frac{1}{4\pi}\frac{x_2(x_2^2+x_1^2)+|x|^3-2x_3^2|x|}{(x_2+|x|)^2|x|^3}$$

$$\frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_2}\left(-\frac{1}{4\pi}\frac{x_3}{(x_2+|x|)|x|}\right) = \frac{1}{4\pi}\frac{x_3}{|x|^3}$$

$$\frac{\partial^2 U_1(x-\xi)}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}\left(-\frac{1}{4\pi}\frac{x_1}{(x_2+|x|)|x|}\right) = \frac{1}{4\pi}\frac{x_1}{|x|^3}.$$

$$\frac{\partial U_1{}^2(x-\xi)}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_3}\left(-\frac{1}{4\pi}\frac{x_1}{(x_2+|x|)|x|}\right) = -\frac{1}{4\pi}\frac{x_1 x_3(x_2+2|x|)}{(x_2+|x|)^2|x|^3} \qquad (3.15)$$

and substituting (315) into (3.14) we establish the following

Theorem 3.2. *If the assumptions of Theorem* 3.1 *hold true then basic relationships* (3.9) *and* (3.14) *are singular.*

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ZADEH'S FUZZY SETS THEORY IN DISTRIBUTION OF CURRENT IN SEMICONDUCTORS

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How is it possible to use Zadeh's Fuzzy sets theory in physical problems? On this question we try answer in this proposal. We consider semiconductor materials that are of great application in our modern technological life.

Today Zadeh's Fuzzy sets theory (FST) takes place great role in applied mathematics, in programming computer and in creating artificial intelligence. It is well known all production of smart technology is based on the semiconductor materials. These materials are object of investigation by physics. Distribution of electrons and holes in semiconductors is very important task and can we use Zadeh's Fuzzy sets theory in our investigation as physicists?

The founder of the fuzzy sets theory, professor L.A.Zadeh, have written in his review to the book entitled "Fuzzy Logic in Chemistry" that he was very astonished when he sew that this theory had found its application even in chemistry. After, he valued this beginning as necessary in development of sciences. Presently there have been made several steps for application of the fuzzy sets theory (FST) also in physics and in our opinion, it seems to be prospective.

The physical science studying the real world phenomena, often comes across uncertainties and therefore make the models of these phenomena that are only copy of the real nature. The types of these models are connected with the evolution of physics. In its early development, physicists mainly used deterministic models. In that period Hartley^s formula was used to represent the uncertainty of the Nonspecifity type. In the next period of its development, physics, having started to consider uncertainties, included in its models probabilistic uncertainty measures of Strife^s type. The statistical physics is a product of that period.

Further, studying the micro-world, physics comes across a quantum uncertainty, Heisenberg principle, based on the probability measure.To consider the probabilistic uncertainty in that period the quantum mechanics was created based on the Bayesian statistical approach.

In our view,the inclusion of the fuzzy uncertainty measure in our models will allow studying many physical phenomena at a higher level of uncertainties hierarchy. To this end the Fuzzy Sets Theory (FST), suggested by L.Zadeh [1],seems to be an adequate mathematical and philosophical technique. The aforesaid is based on the following.

1. A physical model, having been based on a number of assumptions, is approximate and is always characterized by some imprecision. This impreci-

sion is resulted from the conflict between the complexity of the model and the required accuracy. Study of this imprecision, an adequate measure of which is the fuzzy measure, is a subject for FST.

2. Some physical laws are empirical and have been obtained as results of analyses of experimental data [2]. Those data are always supposed to be non-ambiguous, complete and not error-prone. Although, it is almost impossible to carry out the “pure” physical experiments that meet the mentioned features. Therefore, the real world’s phenomena can hardly be described by strict functional relations. Attempts to process such experiment data by statistical methods cannot always be successful due to the four hard assumptions in the statistical regression analysis. In this case the FST can be useful technique to improve the adequacy of empirical laws of physics.
3. Structural analysis of various materials, for example alloys, by using the classical methods is often ineffective in its recognition ability. This inefficiency is associated with the imprecision and uncertainty of the spectrums, i.e. with overlapping of the lines in the spectrums.For this also, methods of FST can be used for determining adequate structures.

The aforesaid ideas can be summarized in the **Table** below.

Summary of general problems of fuzzy physics

	Problems of physics	Supposed methods of solution in fuzzy logic
I	Uncertainty in physics	Possibility theory Fuzzy set theory Fuzzified evidence theory
II	Improvement(perfection) of the classical empirical physical laws	Fuzzy data analysis
III	Interpolation of theoretical regularities	Extension principle of Zadeh
IV	Structure research Structure recognition problems	Fuzzy partition Fuzzy clustering Fuzzy graph theory
V	Analysis of fluctuations and chaos	Fuzzy data analysis Fuzzy differential equations Fuzzy regression analysis Fuzzy computing

There is also one important moment. In physics, the notion of entropy is very basic and it is used in thermodynamics when analyzing chaotic processes. The entropy is the measure of chaos. Fuzziness in chaos, looks like entropy, can be measured by an appropriate fuzzy measure.

Fuzzy sets theory includes also such fuzzy categories as the possibility measure, Π, and necessity measure, N. They are connected with each other as:

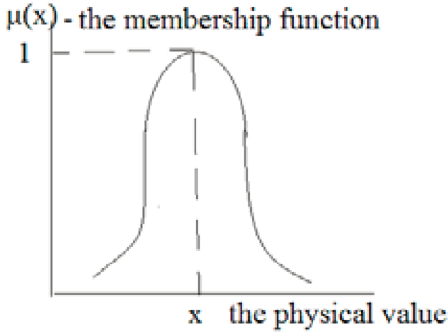
$$W(A) + W(B) \geq 1, \tag{1}$$

$$N(A) + N(B) < 1 \qquad W(A) + N(B) \geq 1 .$$

Where A and **A** ⁻ — are two opposite fuzzy events.

As it is well known, in the probability theory at studying random processes uses the probability measure, P, is used. However, when considering fuzzy processes, the possibility measure, W, and necessity measure can be used, N, too.

The fluctuations are a very interesting problem of physics. In FST the multivariate logic requires introduction of a new function measuring membership grade in the range [0, 1] corresponding to an event. Often the membership fumction is presented as the bell-shaped curves in this Figure.



From this range, “zero” can be corresponded to an event that there are no fluctuations; and “1” can be corresponded to an event that the existing fluctuations are all of the same type (same value and same direction).

The real fluctuations generate unlimited number of fuzzy events and each element from this set can be matched with a value from the range of the membership [0, 1]. So, in our belief, if the fluctuations are of fuzzy nature, then some membership functions, μ, should be involved to study the appropriate processes as necessary.

Of course, one can agree that if the limit theorems of probability theory and statistical theories agree well with experiment, then it would be inappropriate to involve Zade's TNM. But at the same time, we agree that this does not always happen. As noted by many scientists sometimes the discrepancy between the statistically expected and experimental results can even reach 30%. In this case, scientists begin to fight the scatter of the results of the experiment, each in their own way. Physicists pick up soldering irons, chemists clean their reagents, although perhaps there is a more subtle pattern in these scattering.

In our article, we would like to identify this pattern from the point of view of TNM Zadeh. namely, to consider the distribution of minority carriers in the transistors.

As you know, fluctuations are divided into internal and external. Internal fluctuations are fluctuations related to the internal thermodynamic parameters of the system, due to which, for example, the speed of movement of charge carriers will fluctuate. External fluctuations are noise arising from external influences. We will be interested in noise.

It is well known that in any body conducting electricity, due to the chaotic thermal motion of current carriers, voltage fluctuations arise, the time average of which is 0, and the mean square per unit of the frequency band is determined by the Nyquist formula. This effect is called thermal noise. Thermal noise is known to be white noise. It is known that white noise is well studied using stochastic differential equations (SDE). White noise is characterized by the fact that it occurs at all frequencies and the power spectral density F (ω) = const. For thermal noise, the usual thermodynamic derivation of the Nyquist formula assumes that the system is in a state of thermal equilibrium. As many scientists write in the metals, even in the absence of equilibrium, no deviations from the Nyquist formula could be detected. But in semiconductors, the value of the fluctuation voltage between the ends of the crystal when current passes through it can exceed the value given by the Nyquist formula by several orders of magnitude. Even in single crystals and even when measures are taken to eliminate contact effects (that is, this is the case, as described above, when physicists pick up soldering irons again), the magnitude of the fluctuation voltage can be much higher than thermal noise. The nature and origin of such fluctuation noises is of both practical and theoretical interest. They are known to represent, firstly, shot noise, which appears due to spontaneous changes in the concentration of charge carriers, and secondly, the well-known flicker noise (low-frequency noise with a spectral density of 1 / f, where f is the frequency). Thermal and shot noises lend themselves well to study and today there are already good theories explaining their nature, but flicker noise, observed in almost all natural phenomena, but at the same time, being anomalous, still causes a lot of controversy in scientific circles. Even earthquakes are known to be flicker noise. Small in amplitude, every day for a long time before and after the earthquake, the repetitive vibrations of the earth's crust are white noise - they are the result of thermal processes in the earth's crust, but the rare and strong vibrations of the crust after the moment of the earthquake for one or two days are noise of a different nature. It is a flicker noise. This noise is rare, its frequency is low, but it is very strong in amplitude. In electronics, flicker noise is known as excess noise due to the fact that it is much larger than shot and thermal noise.

As you know, flicker noise is observed in almost any electronic device and its sources can be inhomogeneities in a conducting medium, generation and recombination of charge carriers in transistors, etc. One of these transistors are such transistor, which are controlled by a p-n junction. In the appearance of flicker noise in these transistors, the role of the p-n junction, namely in the generation and recombination of charge carriers, is obvious, but how obvious this is and how to prove it using Zade's TNM is the purpose of this article.

Thus, we think that the application of FST in physics has good prospects, and now only first steps are being made. Our first works [4-6] were devoted to consideration of the physical processes taking place in the field of semiconductor and metal contact. This border is connection of two different crystal lattices, and the processes taking place here can be subject of FST study. That work considered the voltage-current characteristics of p-n junction in fuzzy environment.

The p-n junction is the contact of two semiconductors –one with positive (hole)-p, and the other with negative (electron)-n, conductivity. It is know that in the field of p-n tranzition some distribution in the concentrations of current, potential etc. takes place; and this distribution may be fuzzy. In our belief, the application of FST in order to consider these points should be very useful.

To this point the distribution of currency carries in the field of p-n junction is considered below with application of fuzzy sets theory.

The distribution of electrons and holes for energy levels is described by equation [7]:

$$\Delta P\ |_{x=0} = P_n(e^{eu/kT}-1) \qquad (2),$$

Where x is distance from the p-n junction field,
$\Delta P|_{x=0}$ is the surplus concentration of holes in the n-field for x=0;
U is external voltage,
K is Boltzmann constant ($1{,}38\cdot 10^{-23}$ J/K)
E is electron's charge ($1{,}6\cdot 10^{-19}$ coulomb)
As going deeply into the n-field, the concertration of holes decreases following the law:

$$\Delta P_n(x)= \Delta P_n|_{x=0}\, e^{-\frac{x}{L_p}} \qquad (3)$$

As know [8], the continuity equation for holes is expressed as follows:

$$\frac{d^2p}{dx^2} = \frac{p-p_n}{L_p^2} \qquad (4)$$

Where P_n is the concentration of holes in n-field,
L_p is diffusion length of holes in n-field.

If P is expressed as a fuzzy function, then (4) turns out to the following differential equation:

$$\frac{d^2\tilde{p}}{dx^2} = \frac{\tilde{p}-p_n}{L_p^2}, \qquad (5)$$

Then in accordance with α-cut method, (5) can be represented in from:

$$\frac{d^2[p_1^\alpha,p_2^\alpha]}{dx^2} = \frac{[p_1^\alpha,p_2^\alpha]}{L_p^2} \qquad (6)$$

The solution of the equation (6) can be represented in the following from:

$$[p_1^\alpha,p_2^\alpha]= p_n\, [A_1^\alpha,A_2^\alpha]\, e^{-\frac{x}{L_p}} + [B_1^\alpha,B_2^\alpha]\, e^{-\frac{x}{L_p}} \qquad (7)$$

Where $[A_1^\alpha,A_2^\alpha]$, $[B_1^\alpha,B_2^\alpha]$ are the interval coefficients to be found.

It follows from the condition of distribution concentration for x→∞, that $[B_1^\alpha,B_2^\alpha]$ is equal to the fuzzy zero. In this work the fuzzy zero is represented as:

$$[B_1^\alpha,B_2^\alpha] = [\,-\beta_b\sqrt{-ln\alpha}\,,\ \beta_b\sqrt{-ln\alpha}\,] \qquad (8)$$

Where β_b is the parameter of the membership function for the fuzzy zero.
Based on the condition of concentration distribution for $x=Z_n$ (where Z_n is the thickness of the layer with charge of the n-field of the p-n junction)

$$[A_1^\alpha, A_2^\alpha] = [(P_n(e^{\frac{eU}{kT}} - 1) - \beta_b\sqrt{-ln\alpha}\, e^{\frac{Z_n}{l_p}})e^{\frac{Z_n}{l_p}},$$
$$(P_n(e^{\frac{eU}{kT}} - 1) + \beta_b\sqrt{-ln\alpha}\, e^{\frac{Z_n}{l_p}})e^{\frac{Z_n}{L_p}}], \qquad (9)$$

Then the solution of the equation (5) can be represented in the form:

$$[P_1^{\alpha}, P_2^{\alpha}] = [(\mathrm{P}_n(1 + (e^{\frac{eU}{kT}} - 1)e^{\frac{x-Z_n}{l_p}} - \beta_b\sqrt{-ln\alpha}\,e^{\frac{x-2Z_n}{l_p}} - \beta_b\sqrt{-ln\alpha}\,e^{\frac{x}{l_p}}), (\mathrm{P}_n(1 + (e^{\frac{eU}{kT}} - 1)e^{\frac{x-Z_n}{l_p}} + \beta_b\sqrt{-ln\alpha}\,e^{\frac{x-2Z_n}{l_p}} + \beta_b\sqrt{-ln\alpha}\,e^{\frac{x}{l_p}})] \qquad (10)$$

The surplus concentration of currency carries should be represented as

$$\widetilde{\Delta P} = \widetilde{P} - P_{\mathrm{n}} \qquad (11) ,$$

Having considered both (10) and (11) the distribution of surplus currency carries can be represented

$$[\,[\Delta P_n^{1\alpha}, \Delta P_n^{2\alpha}]_{x=0}\,] = [\mathrm{P}_n\, e^{\frac{eU}{kT}} - 1) - 2\beta_b\sqrt{-ln\alpha}\,, (\,e^{\frac{eU}{kT}} - 1) + 2\beta_b\sqrt{-ln\alpha}\,e^{\frac{Z_n}{l_p}}] \qquad (12)$$

In the deep n-field the surplus fuzzy concentration of holes will be:

$$\widetilde{\Delta P}(x) = [\Delta P_n^{1\alpha}, \Delta P_n^{2\alpha}]_{x=0} e^{-\frac{x}{l_p}} - (e^{-\frac{x}{l_p}} - e^{\frac{x}{l_p}})[-\beta_b\sqrt{-ln\alpha}\,, \beta_b\sqrt{-ln\alpha}\,] \quad (13) ,$$

$$\widetilde{\Delta P}(x) = \cup_{\alpha}\, \Delta\widetilde{P^{\alpha}(x)} \qquad (14).$$

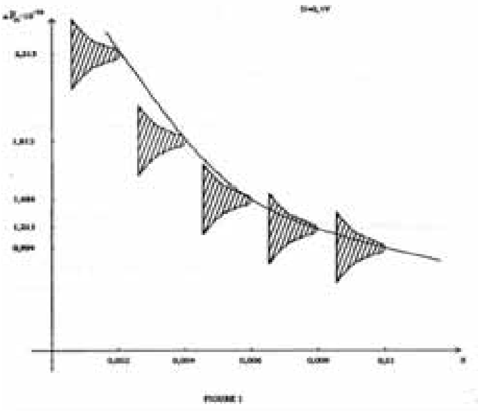
It can be shown that for Ge diodes the uniform concentration of holes is $\mathrm{P_n} = 5.7 * 10^9\ \mathrm{cm}^{-3}$. Then by using the suggested method we can calculate the fuzzy surplus concentration of holes dependent on X and voltage U. For example, in figures 1 and 2 the dependencies of fuzzy surplus concentration of holes on X are shown for the values of voltage u1=0.2V and u2=0.4V, respectively.

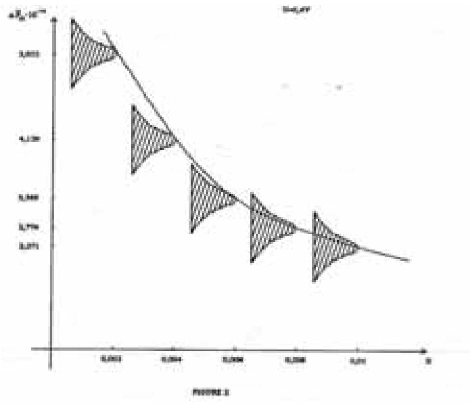
This calculate was made in Azerbaijan State Oil Academy by computer by machine program "Fuzzy Rating" for fuzzy estimates based on FST of Zadeh.

The bell-shaped curves in these Figures 1 and 2 show us the membership function μ, which show us the spread of results in one direction or another. The question arises.

It is known that in an experiment there can be no "ideal" graph, in an experiment there is the same scatter in one direction or another, that is, on both sides, and the graph represents the "average" between this scatter. Doesn't FST of Zadeh tell us about this known us before truth that there must necessarily be a scatter in the results obtained. But this is in experiment, not in theoretical results. In theory, there is the above formula (3), which we took and investigated theoretically from the point of view of TNM Zadeh.

As can be seen from Figures 1 and 2, in both graphs, at large distances X, i.e. far from the border of the pn junction, the exponential curve of the carrier concentration p passes through the middle of the bell-shaped curves of the membership function, breaking it into two parts on either side and the further, the more these parts will be equal to each other, which indicates the same scatter of the results of the carrier concentration charge in both directions.





But at small distances X, i.e. close to the border of the p-n junction, there is a spread in the concentration of charge carriers in one direction, upward. Indeed, in this case, with decreasing X, each subsequent point on the exponential curve is covered by the bell-shaped curve of the membership function of the previous point, and it already has a membership function equal to not 1, but less than 1.

Thus, with the help of Zade's FST, we can once again confirm the correctness of the idea that in field-effect transistors controlled by a p-n junction, the generation and recombination of charge carriers plays an important role in the appearance of flicker noise.

We would like to thank Professor R.A.Aliev from Azerbaijan State Oil Academy for his big helping us in discussion this physical problem.

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THE MOST MAIN IDEA OF LUTFI ZADEH

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In given paper it is being shown why probability theory can not predict such events, named by Nassim Nicolas Taleb as black swans, influence on our life more and more. In paper it has been established that Lutfi Zadeh’s theory of fuzzy sets can be mathematical instrument in such investigation.

Keywords: fuzzy sets, probability, black swan, decision making

On September 6 of 2017 year far from motherland famous scientist Lotfi Zadeh died. The last his wish him was to be being buried in Azerbaijan. President of Azerbaijan Ilham Aliev ordered to organize a worthy funeral for a scientist.

Lotfi Zadeh in his life had got many scientific awards but the most important from them that scientific people become to name him as the father of fuzzy logic. Like Einstein Zade has forced us to look at the world in a new way. Therefore Zade is Einstein in mathematics as say many scientist. It is known there are 2 levels of thinking: the rational thinking and the reasonable thinking. The main difference between these levels of thinking consist of that the reasonable thinking operates with other categories — forms of infinity. Einstein and Zadeh both have used reasonable thinking at creating their scientific theory. They both were right because the world is a thing of infinity. Hence, a logic which includes forms of infinity is necessary for its cognition. But if Einstein have worked in cosmological area dealing with space and time then Zade had deal with uncertainties and he worked out new mathematical theory called theory fuzzy sets (FST).

Now in our world there are so a lot of uncertainties that the problem its sustainable development begin to play important role in life every man. Therefore the humanity must be able to confront such sceneries on world arena. For this confrontation the sustainable development of world is very important and therefore it is necessary to know about its uncertainties. FST Zadeh has deal with uncertainties and therefore it can be used by us to predict and recognize world’s uncertainties. But as it is known there is another theory – theory of probabilities (PT) – that created by scientists before Zadeh and it has deal with uncertainties too. What is difference between these two theories, PT and FST of Zadeh? What is the advantage of FST Zadeh? Let's try to answer this question.

PT is based on Kolmogorov's axiomatic where any event is the set of elementary independent incompatible events with the equal probabilities. For example, the play dice has got six faces. All these faces are elementary events with probabilities equaled 1/6. From these elementary events one can create any events. For example, the event the face of dice to has got 3 or 5 points after test. The probability of this event will equal 1/3=1/6+1/6. For dice it is easy to find the probability of any given event. Because here we see from how many elementary events the given event consists. But it is idealization. In real world we say about probability of given events after tests. “How often occur this event?” is the random value. The randomness is connected with the probability. Really, the random value is the “device” for us to find its probability. Information about randomness give us the distribution of probabilities density. There are many distributions and one of them is a normal distribution named by Gaussian distribution. Knowing density of probability will allow us to say about average of random value. Saying about dice again where probability is 1/3 the average of random of appearance these faces from 60 tests equals 60 * 1/3 = 20 and probability is 1/3 = 20/60 too. Probability will be 1 if it does not matter to us which face of the dice takes place after test. The average of the random value will be 60 * 1 = 60. Analogically, probability is equal 1/6 for one face from all faces and average will be 60 * 1/6 = 10. All these can be shown on Gaussian distribution. And it is truth in case of dice. But let’s ask will it be truth for reality? Can we see elementary events in some event and be sure them to independent? In his famous book “The Black Swan. The Impact of the Highly Improbable”, Nassim Nicholas Taleb has being written that the normal distribution is the “great intellectual lie”. Not in such a strict form, Lutfi Zadeh also criticizes the normal distribution figuratively speaking that if you only have a hammer in your hand, then everything seems to be nails to you. There are many “black swans” that is improbable and uncertain events in our life. The tragic terroristic events in 2001 year on September 11 in USA are “black swans”. It is necessary to be able to predict them. For this aim we must have a good mathematical theory. And we have such theory. This theory is Lutfi Zadeh’s theory of fuzzy sets (FST) . In his “fuzzy sets“, “Decision making in a fuzzy environment” and others [1-3] papers Lutfi Zadeh give us scheme how can be decision made by us in environment of the highly improbably and uncertainty.

Saying about dice again if the probability is 1/6 then the average of random from 60 tests equals 60 * 1/6 = 10. If it is 1/3 then 60 * 1/3 = 20. All these case of dice can be shown well on the Gaussian distribution. Thus the ideal mathematical world and the ideal Gaussian distribution for it.

But in real world is it so ideal? Can we devide in real our world so simple as in mathematical world of dices on elementary independent and incompatible events? Ofcorse, it is not possible.

Therefore Richard von Mises, austrian mathematics, has critised strongly the probability theory. In his book Probability, Statistics, and Truth, first published in German in 1928, to demonstrate of paradox in PT Mises has said simple example. Let's in Germany the football team wins with probability 0,7, but in England with probability 0,8. The events of games occurring in Germany and England are independence and incompatible . Then probability of win is 0,7+0,8=1,5 that is absurd.

Today the investigations of many scientists have demonstrated (for example in Moscow State University Shnoll and etc [5]) that in studying fluctuations of macriscopic phenomena of various behavior (chemical, biological, radiation) there are strange law in their distribution not explained by PT and Gaussian distribution but can be explained by FST Zade in our opinion.

The most main idea of Zade is to see in these fluctuations are the "device" to observe "fuzziness", to say that not only elementary but phokal events, that are depend each with other, are in each event. For such phocal events can we use Aristotle’s law of the excluded third (in Latin language “tertium non datur”, that is, "the third is not given") that is a law of classical logic, consisting in the fact that of two statements - "A" or "not A" - one is necessarily true? Ofcorse, it is not possible.

Thus Zadeh's fuzzy sets theory is more general theory that can consider depended and compatible with each other elementary events. Let's agree that our real world without any modeling, is namely such world, world consisting of dependent and compatible each from other events.

Lotfi Zadeh was born in Azerbaijan, he was Azerbaijani. But as it said the great scientists have not motherland and nationality. They are people of all humanity and their scientific heritage belong all our civilization.

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INVESTIGATION OF LARGE ELASTIC-PLASTIC DEFORMATIONS USING THE LEFT CAUCHY-GREEN TENSOR

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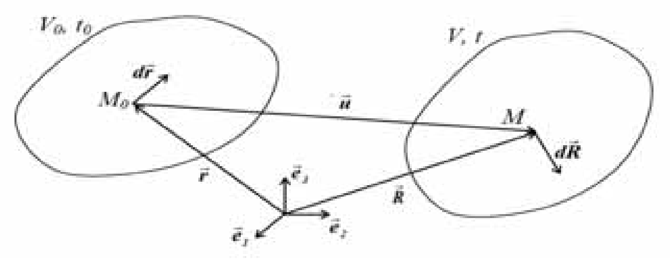
This work is devoted to the development of a technique for studying finite elastic-plastic deformations. A step-by-step loading procedure is used, which is highly algorithmic. The algorithm for constructing a system of linear algebraic equations, as a rule, is the same at each loading step and has a wide range of applicability. Physical relations are determined using the equation of the second law of thermodynamics for isothermal processes, and the equation of the principle of virtual powers in the actual configuration is taken as the basic equation [1,3]. After linearization, a resolving system of linear equations is obtained, where the unknown is the increment of displacements in the current time layer.

Keywords: elastic-plastic deformations, linear algebraic equations, physical relations, the left Cauchy-Green tensor, an elastic-plastic body.

The purpose of this work is to create an algorithm for studying large elastic-plastic deformations and apply it to solving the problems of elastic-plastic deformation of a thick-walled pipe and stretching a round bar.

Let us introduce a global fixed coordinate system with unit vectors $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$. In it, we will consider the positions of the deformable body corresponding to the moment of time t_0 and t.

At the initial moment of time t_0, the position of the body will be called the initial (undeformed state), and at the current moment of time t, the actual (deformed state).



Picture 1. Initial and actual configuration of the deformable body.

Here

V_0 - the volume of the body in the initial configuration;

V - body volume in the current configuration;

$\vec{u}$ - vector of material point displacement;

$\vec{r}$ - radius vector of an arbitrary point M_0 in the initial configuration;

$\vec{R}$ - radius vector of the same point in the current configuration.

We will define radius vectors $\vec{r}$ and $\vec{R}$ as a projection onto global axes $\vec{e_i}$ as follows:

$$\vec{r} = x_i \vec{e_i} \,,$$

$$\vec{R} = y_i \left(x_1, x_2, x_3 \right) \vec{e_i}$$

Let us introduce tensors used in the kinematics of finite deformations.

The first group of tensors describing the kinematics of the medium are strain gradients:

a) strain gradient tensor

$$\left(\vec{\nabla}_x \vec{R} \right)^T = \left(\vec{R} \vec{\nabla}_x \right) = \frac{\partial y_i}{\partial x_j} \left(\vec{e_i} \vec{e_j} \right) = \left(F \right);$$

b) the gradient tensor of place

$$\left(\vec{\nabla}_x \vec{R} \right) = \frac{\partial y_j}{\partial x_i} \left(\vec{e_i} \vec{e_j} \right) = \left(F \right)^T;$$

c) inverse strain gradient tensor

$$\left(\vec{\nabla}_y \vec{r} \right)^T = \left(\vec{r} \vec{\nabla}_y \right) = \frac{\partial x_i}{\partial y_j} \left(\vec{e_i} \vec{e_j} \right) = \left(F^{-1} \right);$$

d) inverse gradient tensor of place

$$\left(\vec{\nabla}_y \vec{r} \right) = \frac{\partial x_j}{\partial y_i} \left(\vec{e_i} \vec{e_j} \right) = \left(F^{-1} \right)^T.$$

Most often, the deformation gradient $\left(F \right)$ is used, which determines the connection of elementary segments in the deformed and undeformed states as follows:

$$d\vec{R} = d\vec{r} \cdot \left(F \right)^T.$$

The second group of tensors are strain measures. It defines:

a) the measure of the Cauchy-Green deformation (the right Cauchy-Green tensor)

$$\left(C \right) = \left(F \right)^T \cdot \left(F \right);$$

b) Finger deformation measure (the left Cauchy-Green tensor)

$$\left(B \right) = \left(F \right) \cdot \left(F \right)^T.$$

In the analysis of the deformation of an elastic-plastic body, the theory of plastic flow is widely used, which is confirmed by direct or indirect two-dimensional or three-dimensional experiments. Based on these experiments, the following conclusions can be drawn.

1. Full deformation in the infinitely close vicinity of a point of the body consists of elastic and plastic parts, and the plastic part is defined as permanent deformation at full unloading.

2. For a wide group of materials, it can be assumed that there is a yield point. For stresses not reaching this limit, the material has a purely elastic behavior. When the yield point is reached, plastic deformation of the body begins.

3. It is assumed that the material has elastic behavior during unloading. It is close to linear.

4. Some materials are considered to be plastically incompressible, that is, they have only elastic volume change.

In this work, a technique is developed for the numerical study of an isotropic material using the left Cauchy-Green tensor. The constitutive relations and the resolving equation are obtained. The algorithm for studying elastic-plastic deformations is applied to solving the problems of elastic-plastic deformation of a thick-walled pipe and tension of a round bar.

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STABILITY OF AN ELASTIC RING UNDER THE ACTION OF A NON-HYDROSTATIC COMPRESSIVE LOAD

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In the presented work, the stability of a long multilayer elastic shell, composed of different materials and subjected to the action of external pressure, has been investigated. The transferred load is not hydrostatic, i.e. it changes significantly in magnitude and direction [1]. It is known that a ring can have a cross-section consisting of several parts connected by concentric circles. When they are rigidly connected, the ring is considered monolithic and can be considered as ordinary, even if its parts are made of different materials.

Keywords: the stability, the elastic shell, the Rayleigh-Ritz method, the control loading parameter, the law, a mixed-type variational method.

Let's define a thin-walled circular ring with radius R, thickness $2h$ and refer it to the polar coordinate system (z,φ) with the origin at the point $z=0$, $-h \le z \le h$, $0 \le \varphi \le 2\pi$. Let us assume that the ring is composed of alternating layers s of different thickness. The thickness of each layer is designated as δ_k. Thus, $\delta_1 + \delta_2 + \ldots + \delta_s = 2h$.

We write the equation of state for the packet as a whole in the form of one equality [2]

$$\varepsilon^{v} = \frac{\sigma}{E_{k+1}}, \qquad a_k \le z \le a_{k+1}, \tag{1}$$

where σ is the stress, and E_{k+1} $[k = 0,1,\ldots,(s-1)]$ is the modulus of elasticity of the material of the k-th layer. In (1), the notation was introduced

$$a_k = -h + \sum_{j=0}^{k} \delta_j, \quad (\delta_0 = 0).$$

Let us now consider the buckling of the selected ring under the action of a compressive load unevenly distributed over the surface of the form - $q = q_0 f(\varphi)$, where $f(\varphi)$ is the given sufficiently smooth function, and q_0 is the control loading parameter.

The solution is based on a mixed-type variational method that takes into account geometric nonlinearity, combined with the Rayleigh-Ritz method [3]. By defining the forms for the function $f(\varphi)$ characterizing the non-hydrostatic loading, and by combining the number of layers in a stack, it is possible to achieve a more efficient and complete use of the bearing capacity of the ring and to control the decrease or increase in the critical force.

Among the problems of stability of thin elastic shells, the problems of stability of cylindrical shells are of the greatest practical importance. This is due to the wide distribution and use of compressed structural elements in various fields of technology and construction. In this section, the problem of the loss stability of a long linear elastic shell of radius R and thickness $2h$. It is assumed that it is compressed by an unevenly distributed radial load, which varies in magnitude and direction according to the law

$$q = q_0\left(1 + \lambda \sin^2 \varphi\right) \tag{2}$$

Neglecting the influence of end fixings, the original problem is reduced to the analysis of the loss of the bearing capacity of a ring of unit width separated from this shell. In this case, various geometrically nonlinear theories are used. Let us denote by v and w, respectively, the displacement in the tangential direction and the deflection. The geometrically nonlinear theory used here is based on the following assumptions:

a) in the process of buckling, nonlinearity in v and w is taken into account simultaneously;

b) neglecting the tangential displacement, we restrict ourselves to the nonlinearity of only the deflection;

c) when $v \approx 0$ the inequality is considered fair $w/R << 1$;

d) the process of stability occurs in the plane of the ring;

e) due to the thinness of the wall, the circumferential stress σ varies in thickness according to a linear law.

Due to the mathematical complexity associated with the need to integrate a nonlinear boundary value problem with variable coefficients, here an approximate solution is carried out by means of a variational method of mixed type. The advantage of this approach is the ability to determine the critical load without solving Euler's differential equations. Moreover, the speed is understood as differentiation with respect to a monotonically increasing parameter q_0, so that

$$\dot{q} = 1 + \lambda \sin^2 \varphi$$

Due to the hypothesis of flat sections, we write

$$\varepsilon = \varepsilon_0 + kz\,,$$

here ε_0 is the initial deformation, and k is the curvature.

Then the expressions for the functionals for theories a), b) and c) have the following form:

$$\begin{aligned} \mathrm{K}_a = R\int_{-h}^{h}\int_{0}^{2\pi}\left\{\dot{\sigma}\dot{\varepsilon} + \frac{\sigma}{2R^2}\left[\left(\frac{\partial \dot{w}}{\partial \varphi} - \dot{v}\right)^2 + \left(\frac{\partial \dot{v}}{\partial \varphi} + \dot{w}\right)^2\right]\right\} d\varphi dz - \\ - \frac{R}{2}\int_{-h}^{h}\int_{0}^{2\pi}\frac{\dot{\sigma}^2}{E} d\varphi dz + R\int_{0}^{2\pi}\dot{q}\dot{w}d\varphi, \end{aligned} \tag{3}$$

$$\mathrm{K}_б = R\int_{-h}^{h}\int_{0}^{2\pi}\left\{\dot{\sigma}\dot{\varepsilon} + \frac{\sigma}{2R^2}\left[\left(\frac{\partial \dot{w}}{\partial \varphi}\right)^2 + \dot{w}^2\right]\right\} d\varphi dz - \frac{R}{2}\int_{-h}^{h}\int_{0}^{2\pi}\frac{\dot{\sigma}^2}{E} d\varphi dz + R\int_{0}^{2\pi}\dot{q}\dot{w}d\varphi, \tag{4}$$

$$K_{в} = R\int_{-h}^{h}\int_{0}^{2\pi}\left\{\dot{\sigma}\dot{\varepsilon} + \frac{\sigma}{2R^2}\left(\frac{\partial \dot{w}}{\partial \varphi}\right)^2\right\}d\varphi dz - \frac{R}{2}\int_{-h}^{h}\int_{0}^{2\pi}\frac{\dot{\sigma}^2}{E}d\varphi dz + R\int_{0}^{2\pi}\dot{q}\dot{w}d\varphi \, , \tag{5}$$

To obtain the final form of the functional, we will use the Rayleigh-Ritz method. For this purpose, as approximating functions we put

$$w = w_0(q) + w_1(q)\cos 2\varphi, \quad v = v_0(q)\sin 2\varphi, \quad M = m(q)\cos 2\varphi \tag{6}$$

or at speeds

$$\dot{w} = \dot{w}_0 + \dot{w}_1\cos 2\varphi \, , \quad \dot{v} = \dot{v}_0\sin 2\varphi \, , \quad \dot{M} = \dot{m}\cos 2\varphi \, . \tag{7}$$

Following assumption e), in the further reasoning we will take

$$\sigma = -\frac{qR}{2h} + \frac{3z}{2h^3}M \quad \text{or} \quad \dot{\sigma} = -\frac{\dot{q}R}{2h} + \frac{3z}{2h^3}\dot{M} \, . \tag{8}$$

The further course of calculations is that expressions for ε are substituted in (3), (4) and (5). Then, in the result obtained, formulas (6), (7), (8) are taken into account. Thus, analytical expressions for the functionals are obtained. Further, the solution can be derived from the stationarity condition $\delta K = 0$ with an additional requirement

$$\frac{dq_0}{dw} = 0 \, .$$

Formulas for the critical force q_0^v are obtained for cases a), b) and c)

$$q_{0(a)}^v = \frac{32}{3(2+\lambda)}\frac{h^3}{R^3}E \, , \quad q_{0(б)}^v = \frac{128}{15(2+\lambda)}\frac{h^3}{R^3}E \, , \quad q_{0(в)}^v = \frac{16}{3(2+\lambda)}\frac{h^3}{R^3}E \, . \tag{9}$$

Hence it can be seen that the parameter λ enters into formulas (9) in the same way. At $\lambda = 0$, the known formulas for the critical hydrostatic load are obtained.

Analyzing formula (9), we can conclude that the influence of different theories on the value of the critical force q_0^v is very significant:

$$q_{0(a)}^v > q_{0(б)}^v > q_{0(в)}^v \, .$$

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